Determinants

Determinants arise very often in linear algebra. Their use is *towards* investigating properties of the matrices and not as a terminal condition. In other words, a determinant is a single number of barely any significance if used in itself but of colossal importance when applying it to the process of finding eigenvalues or the definiteness of symmetric matrices (both topics examined in later sections).

Determinants can only be calculated for square matrices.

However, it only seems natural to start off, with a square matrix of 2×2.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The *determinant* of A is simply the quantity $ad - bc$ and is denoted by *det* $A = ad - bc$. Sometimes, it is convenient to write the determinant of A as $|A|$ instead of *det* A. You can use whichever form is more convenient to you.

However, for matrices of higher dimensions the procedure to calculate the determinant is a little bit more complicated.

Cofactor Expansion

$$\text{Let } A = \begin{bmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1n} \\ \cdot & \cdot & a_{(i-1)j} & a_{(i-1)(j+1)} & \cdot \\ \cdot & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \cdot \\ \cdot & a_{(i+1)(j-1)} & a_{(i+1)j} & \cdot & \cdot \\ a_{n1} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}$$

Before giving the formula for the determinant of an n×n matrix A, define:
(i) A_{ij} as the sub-matrix obtained by deleting the ith row and the jth column,
(ii) the $(\mathbf{i},\mathbf{j})-$**cofactor** of A, as the number C_{ij} given by $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Also note that the determinant of an $n \times n$ matrix A can be computed by a cofactor expansion *across any row or down any column.*

For example, the cofactor expansion across the ith row is *det* $A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$. On the other hand, the cofactor expansion across the jth column is *det* $A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$. Of course, whichever row or column you use, the determinant will be the same. However, it is of more convenience to use the row or the column with the greater number of zeros unless there are no zeros in the matrix.

Example

Compute the determinant of $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$.

Observe that the first row contains two zeros, therefore I will use cofactor expansion across that first row. Hence, given that we choose $i = 1$, $det\ A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ where $C_{1j} = (-1)^{1+j} \det A_{1j}$ for $j = 1, 2, 3$.

Thus,

$$det\ A = (1)(-1)^{1+1} \det \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} + \underbrace{(0)(-1)^{1+2} \det \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} + (0)(-1)^{1+3} \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}}_{=0} = 2$$

Example

Compute $det\ A$, where $A = \begin{bmatrix} 4 & -7 & 8 & 6 \\ 0 & -3 & -4 & -4 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 3 & 1 \end{bmatrix}$.

Observe that the first column contains the most zeros, therefore to make our life easier it is pertinent to use cofactor expansion across that first column. Hence, given that we choose $j = 1$ then $det\ A = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}$ where $C_{i1} = (-1)^{i+1} \det A_{i1}$ for $i = 1, 2, 3, 4$.

$$\begin{aligned} det\ A &= (4)(-1)^{1+1} \det \begin{bmatrix} -3 & -4 & -4 \\ 0 & 2 & 8 \\ 0 & 3 & 1 \end{bmatrix} + \underbrace{(0)(-1)^{2+1} \det A_{21} + (0)(-1)^{3+1} \det A_{31} + (0)(-1)^{4+1} \det A_{41}}_{=0} \\ &= (4)(-1)^{1+1} \det \begin{bmatrix} -3 & -4 & -4 \\ 0 & 2 & 8 \\ 0 & 3 & 1 \end{bmatrix} \end{aligned}$$

We still need to calculate the $det \begin{bmatrix} -3 & -4 & -4 \\ 0 & 2 & 8 \\ 0 & 3 & 1 \end{bmatrix}$.

Again, cleverly we choose the first column with the two zeros.

$$det \begin{bmatrix} -3 & -4 & -4 \\ 0 & 2 & 8 \\ 0 & 3 & 1 \end{bmatrix}$$

$$= (-3)(-1)^{1+1} \det \begin{bmatrix} 2 & 8 \\ 3 & 1 \end{bmatrix} + \underbrace{(0)(-1)^{2+1} \det A_{21} + (0)(-1)^{3+1} \det A_{31}}_{=0}$$

$$= (-3)(-1)^{1+1} \det \begin{bmatrix} 2 & 8 \\ 3 & 1 \end{bmatrix} = (-3)(-1)^{1+1}[(2)(1) - (8)(3)] = 66$$

Therefore, $det\ A = (4)(-1)^{1+1} \det \begin{bmatrix} -3 & -4 & -4 \\ 0 & 2 & 8 \\ 0 & 3 & 1 \end{bmatrix} = (4)(-1)^{1+1}(66)$=264