Eigenvectors, Eigenvalues and Eigenspace

Suppose, matrix A is of $n \times n$ dimension and the nonzero vector $\mathbf{x}$ lives in $\Re^n$. Then if $A\mathbf{x} = \lambda\mathbf{x}$, the scalar λ is called an **eigenvalue** of A and $\mathbf{x}$ is called an **eigenvector** corresponding to eigenvalue λ.

An equivalent definition, is to state that λ is an eigenvalue of A if and only if, the equation $(A - \lambda I)\mathbf{x} = 0$ has a nontrivial solution. In other words, the set of all solutions of $(A - \lambda I)\mathbf{x} = 0$, is just the null space of the matrix $A - \lambda I$. This set is a subspace of $\Re^n$ and is called the **eigenspace** of A corresponding to λ. The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

It is straightforward to determine if a given vector is an eigenvector of a matrix as well as if a specified scalar is an eigenvalue.

Example

Let $A = \begin{bmatrix} 2 & -1 & -4 \\ 3 & 5 & 9 \\ 4 & -2 & 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$.

Are $\mathbf{v}$ and $\mathbf{w}$ eigenvectors of A?

$$A\mathbf{v} = \begin{bmatrix} 2 & -1 & -4 \\ 3 & 5 & 9 \\ 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ -6 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} = 3\mathbf{v}.$$

Thus $\mathbf{v}$ is an eigenvector corresponding to an eigenvalue 3.

$$A\mathbf{w} = \begin{bmatrix} 2 & -1 & -4 \\ 3 & 5 & 9 \\ 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 20 \\ -17 \\ 4 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

Thus, $\mathbf{w}$ is not an eigenvector of A, because $A\mathbf{w}$ is not a multiple of $\mathbf{w}$.