Exercises

1. Consider the subspace $W \subset \Re^4$defined as

$$W = span\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -2 \\ 2 \end{bmatrix} \}.$$

Perform the Gram-Schmidt process to obtain an orthonormal basis for W.

Solution

Recall that $w_n' = v_n - \sum_{i=1}^{n-1} \frac{<v_n, w_i>}{w_i} w_i$

$$w_1 = \frac{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}}{\sqrt{< \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} >}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$w_2' = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - < \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} > \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$w_2 = \frac{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}}{\sqrt{< \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} >}} = \frac{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

w_3'

$$= \begin{bmatrix} -3 \\ 2 \\ -2 \\ 2 \end{bmatrix} - < \begin{bmatrix} -3 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} > \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$- < \begin{bmatrix} -3 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} > \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 2 \\ -2 \\ 2 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2.5 \\ 0 \\ -2 \\ 2.5 \end{bmatrix}$$

$$w_2 = \frac{\begin{bmatrix} -2.5 \\ 0 \\ -2 \\ 2.5 \end{bmatrix}}{\sqrt{< \begin{bmatrix} -2.5 \\ 0 \\ -2 \\ 2.5 \end{bmatrix}, \begin{bmatrix} -2.5 \\ 0 \\ -2 \\ 2.5 \end{bmatrix} >}} = \frac{\begin{bmatrix} -2.5 \\ 0 \\ -2 \\ 2.5 \end{bmatrix}}{\sqrt{16.5}} \begin{bmatrix} -0.615 \\ 0 \\ -0.492 \\ 0.615 \end{bmatrix}$$

2. Find the determinant of the following matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 7 & 0 & 0 & 0 & 0 \\ 6 & 0 & \pi & 4 & e \\ -3 & 0 & 1 & 2 & x \\ 0 & 0 & 1 & x & -1 \end{bmatrix}$

Solution

Observe that the second column contains the most zeros, therefore to make our life easier it is pertinent to use cofactor expansion across that second column. Hence, given that we choose $j = 2$ then $det\ A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} + a_{42}C_{42} + a_{52}C_{52}$ where $C_{i2} = (-1)^{i+2} \det A_{i2}$ for $i = 1, 2, 3, 4, 5$.

Recall that A_{ij} is the matrix obtained by deleing the ith row and the jth column.

det A

$$= (2)(-1)^{1+2} \det \begin{bmatrix} 7 & 0 & 0 & 0 \\ 6 & \pi & 4 & e \\ -3 & 1 & 2 & x \\ 0 & 1 & x & -1 \end{bmatrix}$$

$$+\underbrace{(0)(-1)^{2+2} \det A_{22} + (0)(-1)^{3+2} \det A_{32} + (0)(-1)^{4+2} \det A_{42} + (0)(-1)^{5+2} \det A_{52}}_{=0}$$

$$= (2)(-1)^{1+2} \det \begin{bmatrix} 7 & 0 & 0 & 0 \\ 6 & \pi & 4 & e \\ -3 & 1 & 2 & x \\ 0 & 1 & x & -1 \end{bmatrix}$$

I still need to calculate, $\det \begin{bmatrix} 7 & 0 & 0 & 0 \\ 6 & \pi & 4 & e \\ -3 & 1 & 2 & x \\ 0 & 1 & x & -1 \end{bmatrix}$.

Observe that the first row contains three zeros, therefore I will use cofactor expansion across that first row. Hence, given that we choose $i = 1$, *det* $A = a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}+a_{14}C_{14}$ where $C_{1j} = (-1)^{1+j} \det A_{1j}$ for $j = 1, 2, 3, 4$.

Thus,

$$\det \begin{bmatrix} 7 & 0 & 0 & 0 \\ 6 & \pi & 4 & e \\ -3 & 1 & 2 & x \\ 0 & 1 & x & -1 \end{bmatrix}$$

$$= (7)(-1)^{1+1} \det \begin{bmatrix} \pi & 4 & e \\ 1 & 2 & x \\ 1 & x & -1 \end{bmatrix} + \underbrace{(0)(-1)^{1+2} \det A_{12} + (0)(-1)^{1+3} \det A_{13} + (0)(-1)^{1+4} \det A_{14}}_{=0}$$

$$= (7)(-1)^{1+1} \det \begin{bmatrix} \pi & 4 & e \\ 1 & 2 & x \\ 1 & x & -1 \end{bmatrix}$$

Now, I need to calculate the $\det \begin{bmatrix} \pi & 4 & e \\ 1 & 2 & x \\ 1 & x & -1 \end{bmatrix}$.

Since there are no zeros, I will use cofactor expansion across the first column. Hence, given that we choose $j = 1$ then $det\ A = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$ where $C_{i1} = (-1)^{i+1} \det A_{i1}$ for $i = 1, 2, 3$.

$$= (\pi)(-1)^{1+1} \det \begin{bmatrix} 2 & x \\ x & -1 \end{bmatrix} + (1)(-1)^{2+1} \det \begin{bmatrix} 4 & e \\ x & -1 \end{bmatrix} + (1)(-1)^{3+1} \det \begin{bmatrix} 4 & e \\ 2 & x \end{bmatrix}$$

$$= (\pi)(-2 - x^2) + (4 + ex) + (4x - 2e)$$

Going forward,

$$det\ A = (2)(-1)^{1+2} \det \begin{bmatrix} 7 & 0 & 0 & 0 \\ 6 & \pi & 4 & e \\ -3 & 1 & 2 & x \\ 0 & 1 & x & -1 \end{bmatrix} = (2)(-1)^{1+2}(7)(-1)^{1+1} \det \begin{bmatrix} \pi & 4 & e \\ 1 & 2 & x \\ 1 & x & -1 \end{bmatrix}$$

$$= (2)(-1)^{1+2}(7)(-1)^{1+1}[(\pi)(-2 - x^2) + (4 + ex) + (4x - 2e)]$$

$$= (-14)[(-2\pi - \pi x^2) + (4 + ex) + (4x - 2e)]$$

$$= 28\pi + 14\pi x^2 - 56 - 14ex - 56x + 28e$$

3. Let A be defined as $A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$

a. Find the eigenvalues λ_1, λ_2 of A.
b. For each eigenvalue find an associated eigenvector $v\lambda_1, v\lambda_2$.

c. Write the vector $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ as a linear combination of the eigenvectors. In other words, find scalars x_1 and x_2 such that $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = x_1 v\lambda_1 + x_2 v\lambda_2$.

d. Compute $\lim\limits_{n\to\infty} A^n \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

Solution

a. $det(\begin{bmatrix} 0.6 - \lambda & 0.3 \\ 0.4 & 0.7 - \lambda \end{bmatrix} = \lambda^2 - 1.3\lambda + 0.3 = (\lambda - 1)(\lambda - 0.3) \implies \lambda_1 = 1$ and $\lambda_2 = 0.3$

b. Use characteristic equation $(A - \lambda I)\vec{x} = \vec{0}$

$\lambda_1 = 1$

$$\begin{bmatrix} -0.4 & 0.3 & 0 \\ 0.4 & -0.3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{3}{4} & 0 \\ 1 & -\frac{3}{4} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{3}{4}x_2$
$x_2 = \alpha$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} \Leftrightarrow span \left\{ \begin{matrix} \frac{3}{4} \\ 1 \end{matrix} \right\} \text{Hence } v\lambda_1 = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

$\lambda_2 = 0.3$

$$\begin{bmatrix} 0.3 & 0.3 & 0 \\ 0.4 & 0.4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$
$x_2 = \alpha$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Leftrightarrow span \left\{ \begin{matrix} -1 \\ 1 \end{matrix} \right\} \text{Hence } v\lambda_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

FYI: There exists a Theorem that states that if $\mathbf{v}_1, \cdots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ is linearly independent.

c. $\begin{bmatrix} v\lambda_1 & v\lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

$$\begin{bmatrix} \frac{3}{4} & -1 & 0.5 \\ 1 & 1 & 0.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0.5 \\ \frac{3}{4} & -1 & 0.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & -\frac{7}{4} & \frac{1}{8} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{4}{7} \\ 0 & 1 & -\frac{1}{14} \end{bmatrix}$$

Verify that $\frac{4}{7} \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} + \frac{-1}{14} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

d. $\lim_{n\to\infty} A^n \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \lim_{n\to\infty} A^n(x_1 v\lambda_1 + x_2 v\lambda_2) = \lim_{n\to\infty} A^n x_1 v\lambda_1 + \lim_{n\to\infty} A^n x_2 v\lambda_2 =$
$x_1 \lim_{n\to\infty} A^n v\lambda_1 + x_2 \lim_{n\to\infty} A^n v\lambda_2 = x_1 \lim_{n\to\infty} \lambda^n v\lambda_1 + x_2 \lim_{n\to\infty} \lambda^n v\lambda_2 = x_1 \lim_{n\to\infty} 1^n \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} +$
$x_2 \lim_{n\to\infty} 0.3^n \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (\frac{4}{7})(1) \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} + (\frac{-1}{14})(0) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}$

4. Consider the matrix $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$

a. Find the eigenvalues of A.
b. Find a basis for the eigenspace of each of the eigenvalues of A.
c. Diagonalize A by finding a matrix P such that $A = PDP^{-1}$.
d. Compute P^{-1}.
e. Compute $exp(A)$.

Solution

a. To find eigenvalues we take the $det(A - \lambda I)$

$$det(\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix})$$

$$= \det(\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}) = \det(\begin{bmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 0 & 0 \\ 1 & 4 & -3-\lambda & 0 \\ -1 & -2 & 0 & -3-\lambda \end{bmatrix})$$

$$= (\lambda - 5)(\lambda - 5)(\lambda + 3)(\lambda + 3)$$

The eigenvalues are $\lambda_1 = \lambda_2 = 5$ and $\lambda_3 = \lambda_4 = -3$

b. Use characteristic equation $(A - \lambda I)\vec{x} = \vec{0}$

$\lambda_1 = \lambda_2 = 5$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 & 0 \\ -1 & -2 & 0 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 8 & 0 \\ 0 & 2 & -8 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 8 & 16 & 0 \\ 0 & 1 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -8x_3 - 16x_4 = -8\alpha - 16\beta$
$x_2 = 4x_3 + 4x_4 = 4\alpha + 4\beta$
$x_3 = \alpha$
$x_4 = \beta$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow span \left\{ \begin{matrix} -8 \\ 4 \\ 1 \\ 0 \end{matrix} , \begin{matrix} -16 \\ 4 \\ 0 \\ 1 \end{matrix} \right\}$$

$\lambda_3 = \lambda_4 = -3$

$$\begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_2 = 0$
$x_3 = \alpha$
$x_4 = \beta$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow span \left\{ \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix} , \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} \right\}$$

c. $P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

d. $$\begin{bmatrix} -8 & 16 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -16 & 8 & 0 & 1 & 0 & 8 & 0 \\ 0 & 4 & -4 & 0 & 0 & 1 & -4 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 8 & 16 & 1 & 0 & 8 & 16 \\ 0 & 0 & -4 & -4 & 0 & 1 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & -4 & 0 & 1 & -4 & -4 \\ 0 & 0 & 0 & 8 & 1 & 2 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -4 & 0 & \frac{1}{2} & -3 & -4 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{8} & \frac{1}{4} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{8} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{8} & \frac{1}{4} & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{4} & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 & 1 \end{bmatrix}$$

It's a good idea to verify that the matrices are correctly calculated.

$A = PDP^{-1}$

$$= \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{4} & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$$

e. $exp(A) = P \begin{bmatrix} e^{d_1} & 0 & 0 & 0 \\ 0 & e^{d_2} & 0 & 0 \\ 0 & 0 & e^{d_3} & 0 \\ 0 & 0 & 0 & e^{d_4} \end{bmatrix} P^{-1}$