Least-Squares Solution

Having defined the concepts of inner product and distance, we are in a position to address the main topic; locating a least-squares solution to a linear system which is inconsistent. Recall from earlier discussions that an acceptable substitute for a solution to the system, is a vector $\hat{\mathbf{x}}$ that makes the distance between $A\hat{\mathbf{x}}$ and $\mathbf{b}$ minimal.

Think of $A\mathbf{x}$ as an approximation to $\mathbf{b}$. The smaller the distance between $\mathbf{b}$ and $A\mathbf{x}$, given by $\|\mathbf{b}-A\mathbf{x}\|$, the better the approximation. The general least-squares solution is to find an $\mathbf{x}$ that makes $\|\mathbf{b}-A\mathbf{x}\|$ as small as possible. The term least-squares arises from the fact that $\|\mathbf{b}-A\mathbf{x}\|$ is the square root of a sum of squares.

If A is an $m \times n$ matrix and $\mathbf{b}$ is in $\Re^m$, a **least-squares solution** of $A\mathbf{x} = \mathbf{b}$ is an $\hat{\mathbf{x}}$ in $\Re^n$ such that $\|\mathbf{b}-A\hat{\mathbf{x}}\| \leq \|\mathbf{b}-A\mathbf{x}\|$ for all $\mathbf{x}$ in $\Re^n$.

The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ coincides with the nonempty set of solutions of the normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$.

Example

Find all least squared solutions to the following problem.

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Step 1: Show that the system is inconsistent

$$\begin{bmatrix} -1 & 2 & 4 \\ 2 & -3 & 1 \\ -1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 9 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 14 \\ 0 & 1 & 9 \\ 0 & 0 & -11 \end{bmatrix}$$

The system is inconsistent since there is a pivot column in the rightmost column of the augmented matrix.

Step 2: Recall that $A^T A\hat{x} = A^T\beta$

$$A^T A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

$$A^T\beta = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix} \hat{x} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & \frac{6}{11} \end{bmatrix} \begin{bmatrix} -4 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \text{Hence } \hat{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$