Linear Independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ in $\Re^n$ is said to be **linearly independent** if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_p\mathbf{v}_p = 0$ has only the trivial solution $(x_1 = x_2 = \cdots = x_p)$. Otherwise, if this condition fails the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ is said to be **linearly dependent.**

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

(a) Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

If there exists a non-trivial solution, then the system is linearly independent. This translates into having three basic variables. Therefore we need to row reduce the homogeneous augmented matrix.

$$\begin{bmatrix} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 15 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly, x_1 and x_2 are basic variables and x_3 is free. Each nonzero value of x_3 determines a nontrivial solution. Therefore the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

Sets of One or Two Vectors

A set containing only one vector is linearly independent unless it is the zero vector. A set of two vectors $\{v_1, v_2\}$ is linearly dependent if and only if one of the vectors is a multiple of the other.

Sets of Two or More Vectors

If a set contains more vectors than there are entries in each vector, the set is linearly dependent. That is, any set $\{v_1, v_2, \cdots, v_p\}$ in $\Re^n$, is linearly dependent if $p > n$. If a set $\{v_1, v_2, \cdots, v_p\}$ in $\Re^n$ contains the zero vector, then the set is linearly dependent.

When calculating determinants, the matrix *has* to be a square matrix.