Orthogonal Vectors

Consider two lines in $\Re^2$ through the origin determined by vectors $\mathbf{u}$ and $\mathbf{v}$. The two lines are geometrically *perpendicular (orthogonal)* if and only if the line through $\mathbf{u}$ is the perpendicular bisector of the line segment from $-\mathbf{v}$ to $\mathbf{v}$. This is the same as requiring the squares of the distances from $\mathbf{u}$ to $-\mathbf{v}$ and from $\mathbf{u}$ to $\mathbf{v}$ to be equal.

Thus,
$[dist(\mathbf{u}, -\mathbf{v})]^2 = \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$
$=< (\mathbf{u} + \mathbf{v}), (\mathbf{u} + \mathbf{v}) >$
$=< \mathbf{u}, (\mathbf{u} + \mathbf{v}) > + < \mathbf{v}, (\mathbf{u} + \mathbf{v}) >$
$=< \mathbf{u}, \mathbf{u} > + < \mathbf{u}, \mathbf{v} > + < \mathbf{v}, \mathbf{u} > + < \mathbf{v}, \mathbf{v} >$
$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2 < \mathbf{u}, \mathbf{v} >$

The same calculations with $\mathbf{v}$ and $-\mathbf{v}$ interchanged show that
$[dist(\mathbf{u}, \mathbf{v})]^2 = \|\mathbf{u} - \mathbf{v}\|^2$
$=< (\mathbf{u} - \mathbf{v}), (\mathbf{u} - \mathbf{v}) >$
$=< \mathbf{u}, (\mathbf{u} - \mathbf{v}) > + < -\mathbf{v}, (\mathbf{u} - \mathbf{v}) >$
$=< \mathbf{u}, \mathbf{u} > + < \mathbf{u}, -\mathbf{v} > + < -\mathbf{v}, \mathbf{u} > + < -\mathbf{v}, -\mathbf{v} >$
$= \|\mathbf{u}\|^2 + \|-\mathbf{v}\|^2 + 2 < \mathbf{u}, -\mathbf{v} >$
$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2 < \mathbf{u}, \mathbf{v} >$

The two squared distances are equal if and only if $2 < \mathbf{u}, \mathbf{v} >= -2 < \mathbf{u}, \mathbf{v} >$, which happens if and only if $< \mathbf{u}, \mathbf{v} >= 0$.

This calculation shows that when vectors $\mathbf{u}$ and $\mathbf{v}$ are identified with geometric points, the corresponding lines through the points and the origin are perpendicular if and only if $< \mathbf{u}, \mathbf{v} >= 0$.

The following definition generalizes to $\Re^n$ this notion of perpendicularity (orthogonality).

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\Re^n$ are **orthogonal** (to each other) if and only if:
$< \mathbf{u}, \mathbf{v} >= 0$