Significance of the LU Decomposition

The short answer is that the LU Factorization is faster (in terms of floating point operations (flops)) when solving for $\mathbf{x}$, than row reduction of $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ to $\begin{bmatrix} I & \mathbf{x} \end{bmatrix}$.

In specific, the significance of this decomposition arises when we transform the equation $A\mathbf{x} = \mathbf{b}$ into $L(U\mathbf{x}) = \mathbf{b}$. Writing $\mathbf{y}$ for $U\mathbf{x}$, we can find $\mathbf{x}$ by solving the pair of equations:
$L\mathbf{y} = \mathbf{b}$
$U\mathbf{x} = \mathbf{y}$

Outline of the Procedure:
1. Solve $L\mathbf{y} = \mathbf{b}$,
2. Solve $U\mathbf{x} = \mathbf{y}$.

Note the usefulness of the decomposition in that L and U are triangular.

Example

It has been verified in the previous example that,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{1}{6} & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 12 & 3 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 8 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Use this LU factorization to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix}$.

$$[L \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & 2 \\ -1 & -\frac{1}{6} & 1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix} = [I \quad \mathbf{y}]$$

$$[U \quad \mathbf{y}] = \begin{bmatrix} 1 & -2 & -1 & -2 \\ 0 & 12 & 3 & 6 \\ 0 & 0 & \frac{1}{2} & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 10 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 4 \\ -2 \\ 10 \end{bmatrix}.$$

Observe, that the LU Decomposition of the matrix A does in fact, require fewer flops when solving for $\mathbf{x}$, than the row reduction procedure.