Solving a Linear System

A system of linear equations can be conveniently entered in a rectangular array called **matrix**. The procedure is simple; just isolate the coefficients from the variables and record the coefficients in an orderly fashion into a **coefficient matrix**. Given the system

$$\begin{aligned} 2x_1 - 2x_2 + 8x_3 &= 10 \\ 4x_1 - 6x_2 + 2x_3 &= 4 \\ 2x_2 + 4x_3 &= 6 \end{aligned}$$

the coefficient matrix is $\begin{bmatrix} 2 & -2 & 8 \\ 4 & -6 & 2 \\ 0 & 2 & 4 \end{bmatrix}$.

In addition, one can also incorporate into the coefficient matrix the constants from the right side of the equations.

The **augmented matrix** would then be $\begin{bmatrix} 2 & -2 & 8 & 10 \\ 4 & -6 & 2 & 4 \\ 0 & 2 & 4 & 6 \end{bmatrix}$.

The augmented matrix has dimensions 3×4 (read "3 by 4"). It indicates that there are 3 rows and 4 columns. It is a convention in linear algebra to read the rows first and then the columns. More generally, an mxn matrix comprises of m rows and n columns. Oftentimes a matrix with the same number of rows as columns is called a **square** matrix for the obvious reason.

To solve the system, one needs to substitute the current system with an equivalent one that is easier to solve. In this quest, three elementary row operations will be used extensively:
(i)multiplication of all entries in a row by a nonzero constant,
(ii)interchangeability of two rows,
(iii)replacement of one row by the sum of itself and a multiple of another row.

In the following example these elementary operations are demonstrated. With practice, you will be able to perform row operations at no time. Consequently determining whether a system is consistent or not and if it is in fact consistent, whether it has a unique solution or a solution set, will be done relatively easy, at least, for low-dimension augmented matrices.

Consider again

$$\begin{aligned} 2x_1 - 2x_2 + 8x_3 &= 10 \\ 4x_1 - 6x_2 + 2x_3 &= 4 \\ 2x_2 + 4x_3 &= 6 \end{aligned}$$

that leads to the augmented matrix $\begin{bmatrix} 2 & -2 & 8 & 10 \\ 4 & -6 & 2 & 4 \\ 0 & 2 & 4 & 6 \end{bmatrix}$.

Applying the above elementary operations will give us the unique solution of this system.

$$\begin{bmatrix} 2 & -2 & 8 & 10 \\ 4 & -6 & 2 & 4 \\ 0 & 2 & 4 & 6 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 4 & 5 \\ 4 & -6 & 2 & 4 \\ 0 & 2 & 4 & 6 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 4 & 5 \\ 0 & -2 & -14 & -16 \\ 0 & 2 & 4 & 6 \end{bmatrix}$$

$$\overset{3}{\sim} \begin{bmatrix} 1 & -1 & 4 & 5 \\ 0 & 2 & 4 & 6 \\ 0 & -2 & -14 & -16 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & -1 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -14 & -16 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & -1 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -10 & -10 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & -1 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\overset{7}{\sim} \begin{bmatrix} 1 & 0 & 6 & 8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{8}{\sim} \begin{bmatrix} 1 & 0 & 6 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{9}{\sim} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Analytically:

$\overset{1}{\sim}$:= multiply the first row by $\frac{1}{2}$,

$\overset{2}{\sim}$:= replace the second row by the sum of itself and the first row when the latter is multiplied by -4,

$\overset{3}{\sim}$:= interchange the second row with the third,

$\overset{4}{\sim}$:= multiply the second row by $\frac{1}{2}$,

$\overset{5}{\sim}$:= replace the third row by the sum of itself and the second row when the latter is multiplied by 2,

$\overset{6}{\sim}$:= multiply the third row by $-\frac{1}{10}$,

$\overset{7}{\sim}$:= replace the first row by the sum of itself and the second row when the latter is multiplied by 1,

$\overset{8}{\sim}$:= replace the second row by the sum of itself and the third row when the latter is multiplied by -2,

$\overset{9}{\sim}$:= replace the first row by the sum of itself and the third row when the latter is multiplied by -6

Thus, $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ signifies that $x_1 = 2$, $x_2 = 1$, $x_3 = 1$.

Why? Remember that our objective when solving any system of linear equations is to reduce it to a very simplified but *equivalent* form. In this most simplified form, the first row can be read as $x_1 + 0x_2 + 0x_3 = 2$, the second row as $0x_1 + x_2 + 0x_3 = 1$ and the third row as $0x_1 + 0x_2 + x_3 = 1$. Thus, very conveniently the system is reduced to the unique solution $x_1 = 2$, $x_2 = 1$, $x_3 = 1$.