A factorization of a matrix A is an equation that expresses A as a product of two or more matrices.

The LU Decomposition

The LU decomposition, is a procedure where a matrix A of $m \times n$ dimension, is decomposed into a *lower triangular matrix* L (with 1's on the diagonal) of dimension $m \times m$, and an *upper triangular matrix* U of dimension $m \times n$. Thus, A is factorized into L and U. Such a decomposition is called the **LU decomposition** of A.

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} LE & * & * & * & * \\ 0 & LE & * & * & * \\ 0 & 0 & 0 & LE & * \\ 0 & 0 & 0 & 0 & LE \end{bmatrix}$$

Note that the starred ($*$) entries can be any real numbers, whereas "LE" defined as $\Re\backslash\{0\}$, stands for the leading entry in the row.

An LU Factorization Algorithm

Suppose A can be reduced to an echelon form U (recall that U is an $m \times n$ echelon form of A) *without* row interchanges. Then since row scaling is *not* essential, A can be reduced to U with *only* row replacements (replacement of one row by the sum of itself and a multiple of another row).

Consequently, there exist *unit lower triangular elementary matrices* $E_1, E_2, \cdots, E_p$ such that $(E_p \cdots E_1)A = U$.

Therefore, $A = IA = (E_p \cdots E_1)^{-1}(E_p \cdots E_1)A = (E_p \cdots E_1)^{-1}U = LU$, if we define $L := (E_p \cdots E_1)^{-1}$.

To summarize:

1. Determine the dimension of U and L based on the dimension of A.
2. Reduce A to an echelon form U by a sequence of *only* row replacement operations, if possible.
3. Place entries in L such that the same sequence of row operations reduces L to I.

Example

Find an LU factorization of $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 8 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

Step 1: Determine the dimension of U and L based on the dimension of A. Since A is a 3×3 matrix, L should be 3×3 and so must U.

Step 2: Reduce A to an echelon form U by a sequence of only row replacement operations.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then,

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 2 & 8 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & \mathit{12} & 3 \\ 0 & -2 & 0 \end{bmatrix};$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{6} & 1 \end{bmatrix}$$

Then,

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{6} & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 12 & 3 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 12 & 3 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = U$$

Hence, $A = L(E_2E_1A)$

Step 3: Place entries in L such that the same sequence of row operations reduces L to I.

$$L = (E_2E_1)^{-1} = E_1^{-1}E_2^{-1}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{6} & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{6} & 1 \end{bmatrix}$$

Therefore,

$$L = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{6} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{1}{6} & 1 \end{bmatrix}$$

Again it is a good idea to verify that L and U satisfy $LU = A$.

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{1}{6} & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 12 & 3 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 8 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$