Appendix

Proposition 1 *(Cauchy-Schwartz inequality)*

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}} \tag{1}$$

Proof. Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(\lambda) = \sum_{i=1}^{n}(y_i - \lambda x_i)^2$.
Expanding, we get

$$f(\lambda) = \underbrace{\sum_{i=1}^{n} y_i^2}_{c} \quad -2\lambda \underbrace{\sum_{i=1}^{n} x_i y_i}_{b} \quad +\lambda^2 \underbrace{\sum_{i=1}^{n} x_i^2}_{a}$$

We consider now two cases: $a > 0$, and $a = 0$.
(i)$a > 0$
In this case $f(\lambda)$ is a parable, and its minimum is at the point λ^* that satisfies $f'(\lambda^*) = 2a\lambda^* - 2b = 0$, so we get $\lambda^* = \frac{b}{a}$.
Notice that $f(\lambda) \geq 0$, since it is a sum of squares. In particular, $f(\lambda^*) = \frac{-b^2+ac}{a} \geq 0$, which implies $ac \geq b^2$. Rewriting this we get

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right)$$

Taking square root on both sides we get the result
(ii)$a = 0$
If $a = 0$, then $x_i = 0$ for all $i = 1, ..., n$, and then the inequality is trivially satisfied since $0 \leq 0 \cdot \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}}$ ∎