

## Metric Spaces

### Definitions

**Definition 1** Let  $S$  be a set and suppose that  $d : S \times S \rightarrow \mathbb{R}$ . We say that  $(S, d)$  is a metric space if  $d$  satisfies the following properties:

- (i)  $d(x, y) \geq 0$ , for all  $x, y \in S$ , with  $d(x, y) = 0$  iff  $x = y$
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in S$
- (iii)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in S$

Property (ii) is usually referred as the symmetry property, while (iii) is usually called the triangular inequality.

**Example 2** Consider  $S = \mathbb{R}^n$ , and the following 4 functions:

- $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$
- $d_2(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$
- $d_3(x, y) = \max_{i=1, \dots, n} |x_i - y_i|$
- $d_4(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Then  $(S, d_i)$  is a metric space for  $i = 1, \dots, 4$ . We will prove that for metrics  $d_2$  and  $d_3$ , the other ones are left as exercise to the reader.

**1.**  $d_2(x, y)$

(i) It is clear that  $d_2(x, y) \geq 0$ , since it is a sum of squares, all of them positive.

Also:

$$\begin{aligned} d_2(x, y) = 0 &\iff \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = 0 \\ &\iff \sum_{i=1}^n (x_i - y_i)^2 = 0 \\ &\iff x_i - y_i = 0 \text{ for all } i \in \{1, \dots, n\} \\ &\iff x = y \end{aligned}$$

(ii) Obvious from the fact that  $(x_i - y_i)^2 = (y_i - x_i)^2$   
 (iii) For this we use the Cauchy-Schwartz inequality, so we have:

$$\begin{aligned}
 d_2(x, z) &= \sqrt{\sum_{i=1}^n (x_i - z_i)^2} \\
 &= \sqrt{\sum_{i=1}^n ((x_i - y_i) + (y_i - z_i))^2} \\
 &= \sqrt{\sum_{i=1}^n (x_i - y_i)^2 + 2 \sum_{i=1}^n (x_i - y_i)(y_i - z_i) + \sum_{i=1}^n (y_i - z_i)^2} \\
 \text{Cauchy-Schwartz} \longrightarrow &\leq \sqrt{\sum_{i=1}^n (x_i - y_i)^2 + 2 \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \sqrt{\sum_{i=1}^n (y_i - z_i)^2} + \sum_{i=1}^n (y_i - z_i)^2} \\
 &= \sqrt{\left( \sqrt{\sum_{i=1}^n (x_i - y_i)^2} + \sqrt{\sum_{i=1}^n (y_i - z_i)^2} \right)^2} \\
 &= \sqrt{\sum_{i=1}^n (x_i - y_i)^2} + \sqrt{\sum_{i=1}^n (y_i - z_i)^2} \\
 &= d(x, y) + d(y, z)
 \end{aligned}$$

2.  $d_3(x, y)$

(i) It is clear that  $d_3(x, y) \geq 0$ , since it is a sum of squares, all of them positive.  
 Also:

$$\begin{aligned}
 d_3(x, y) = 0 &\iff \max_{i=1, \dots, n} |x_i - y_i| = 0 \\
 &\iff |x_i - y_i| = 0 \text{ for all } i \in \{1, \dots, n\} \\
 &\iff x = y
 \end{aligned}$$

(ii) Obvious from the fact that  $|x_i - y_i| = |y_i - x_i|$   
 (iii) We have

$$\begin{aligned}
 d_3(x, z) &= \max_{i=1, \dots, n} |x_i - z_i| \\
 &= \max_{i=1, \dots, n} |(x_i - y_i) + (y_i - z_i)| \\
 &= \max_{i=1, \dots, n} (|x_i - y_i| + |y_i - z_i|) \\
 &\leq \max_{i=1, \dots, n} |x_i - y_i| + \max_{i=1, \dots, n} |y_i - z_i| \\
 &= d(x, y) + d(y, z)
 \end{aligned}$$

**Remark:** When  $S = \mathbb{R}$ ,  $d_1, d_2$  and  $d_3$  are the same and reduce to  $d_i(x, y) = |x - y|$ . Unless otherwise stated, this is the distance function we will be using whenever  $S = \mathbb{R}$ .

**Definition 3** Let  $S$  be a set and suppose that  $(S, d_1)$  and  $(S, d_2)$  are metric spaces. We say that  $d_1$  and  $d_2$  are equivalent iff there exists constants  $c$  and  $C$  such that

$$cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y) \text{ for all } x, y \in S \quad (1)$$

**Example 4** Consider  $S = \mathbb{R}^n$  and the four distance functions introduced in example 2. Then

- $d_1, d_2, d_3$  are equivalent
- $d_4$  is not equivalent to any of the other three.

We will just prove that  $d_1$  and  $d_3$  are equivalent, and that  $d_4$  is not equivalent to  $d_3$ , the others are left as an exercise.

**1. Equivalence of  $d_1$  and  $d_3$**

We need to find the constants  $c$  and  $C$ . For that, we do the following inequalities:

$$\begin{aligned} d_3(x, y) &= \max_{i=1, \dots, n} |x_i - y_i| \\ &\leq \sum_{i=1}^n |x_i - y_i| \\ &= 1 \cdot d_1(x, y) \end{aligned}$$

$$\begin{aligned} d_1(x, y) &= \sum_{i=1}^n |x_i - y_i| \\ &\leq \sum_{i=1}^n \max_{j=1, \dots, n} |x_j - y_j| \\ &= n \cdot \max_{j=1, \dots, n} |x_j - y_j| \\ &= n \cdot d_3(x, y) \end{aligned}$$

So we have that  $c = 1$  and  $C = n$ , and the result follows.

**2. Non-equivalence of  $d_3$  and  $d_4$**

We will prove that there exists no  $C > 0$  such that  $d_3(x, y) \leq Cd_4(x, y)$  for all  $x, y \in \mathbb{R}^n$ . In fact, consider any  $C$ , and the vectors  $x = (C + 1, \dots, C + 1)$ ,  $y = (0, \dots, 0)$ . Then  $d_3(x, y) = C + 1 > C = Cd_4(x, y)$ .