

## Sequences and Convergence

Let  $x_1, x_2, \dots, x_n, \dots$  denote an infinite sequence of elements of a metric space  $(S, d)$ . We use  $\{x_n\}_{n=1}^{\infty}$  (or simply  $\{x_n\}$ ) to denote such a sequence.

**Definition 1** Consider  $x_0 \in S$ . We say that the sequence  $\{x_n\}$  converges to  $x_0$  when  $n$  tends to infinity iff: For all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $d(x_n, x_0) < \epsilon$

We denote this convergence by  $\lim_{n \rightarrow \infty} x_n = x_0$  or simply  $x_n \rightarrow x_0$ .

**Example 2** Consider the sequence  $\{x_n\}$  in  $\mathbb{R}$ , defined by  $x_n = \frac{1}{n}$ . Then  $x_n \rightarrow 0$ .

The way to prove this is standard: fix  $\epsilon > 0$ . We need to find  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $d(x_n, 0) < \epsilon$ . We have  $d(x_n, 0) = |x_n - 0| = |\frac{1}{n}|$ . So it is enough that  $\frac{1}{n} < \epsilon$ , or equivalently  $n > \frac{1}{\epsilon}$ . So choosing  $N > \frac{1}{\epsilon}$  we know that for all  $n > N$ ,  $d(x_n, 0) < \epsilon$ .

The fact that we define the concept of convergence does not imply that every sequence converges. This is illustrated in the next two examples. Let's begin with a remark about what it means for a sequence  $\{x_n\}$  not to converge to  $x_0$ .

**Remark:** To know what the non-convergence of a sequence means, we need to write the negation of the definition of convergence. That reduces to: There exists  $\epsilon > 0$ , such that for all  $N \in \mathbb{N}$ , there exists  $n > N$  such that  $d(x_n, x_0) \geq \epsilon$ . For the ones of you familiar with propositional logic, notice that convergence to  $x_0$  can be written as

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)d(x_n, x_0) < \epsilon$$

Its negation is then given by

$$(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\exists n > N)d(x_n, x_0) \geq \epsilon$$

**Example 3** Consider the sequence  $\{x_n\}$  in  $\mathbb{R}$ , defined by  $x_n = n$ . Then  $x_n$  does not converge to any  $x_0$ .

Consider any  $x_0$  in  $\mathbb{R}$  and  $\epsilon = \frac{1}{2}$  (Notice that when we prove non-convergence, it is enough to find one  $\epsilon > 0$ , and we do not need to do the argument for all  $\epsilon$ ). Then, for any  $N$ , just consider any  $n > \max N, x_0 + 1$ . Then, since  $n > x_0 + 1$ ,  $d(x_n, x_0) = |x_n - x_0| > 1 > \epsilon$ , and the result follows.

**Example 4** Consider the sequence  $\{x_n\}$  in  $\mathbb{R}$ , defined by  $x_n = (-1)^n$ . Then  $x_n$  does not converge.

It is obvious that we only need to check that  $x_n$  does not converge to 1 nor -1.

Let's do it for  $x_0 = 1$ .

Consider  $\epsilon = 1$  and, for any  $N$ , consider an  $n > N$  that is odd, so  $x_n = -1$ . Then  $d(x_n, x_0) = |x_n - x_0| = |-1 - 1| = 2 > \epsilon$ , and the result follows.

The next proposition shows that the limit of a sequence (if it exists) is unique.

**Theorem 5** *If  $\{x_n\}$  in  $(S, d)$  is such that  $x_n \rightarrow x_0$  and  $x_n \rightarrow x'_0$ , then  $x_0 = x'_0$ .*

**Proof.** Fix  $\epsilon > 0$ , by the convergence of  $\{x_n\}$  to  $x_0$  and  $x'_0$  we have that: there exists  $N_1 \in \mathbb{N}$ , such that for all  $n > N_1$ ,  $d(x_n, x_0) < \frac{\epsilon}{2}$  and there exists  $N_2 \in \mathbb{N}$ , such that for all  $n > N_2$ ,  $d(x_n, x'_0) < \frac{\epsilon}{2}$ . By choosing  $N = \max\{N_1, N_2\}$ , we know that if  $n > N$ , then  $n$  will be simultaneously greater than  $N_1$  and  $N_2$ . Then we have, for all  $n > N$ :

$$\begin{aligned} d(x_0, x'_0) &\leq d(x_0, x_n) + d(x_n, x'_0) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

We have then proved that  $d(x_0, x'_0) < \epsilon$  for any  $\epsilon > 0$ . This implies that  $d(x_0, x'_0) = 0$  and, by the first property of distance functions, we have  $x_0 = x'_0$ . ■

The next proposition can be quite useful, it basically states that if convergence to a limit is true for one distance function, it is also true for any equivalent distance. In the context of  $S = \mathbb{R}^N$ , it allows to choose  $d_1, d_2, d_3$  (as defined in example ??) to prove convergence, depending on which is easier in a particular context.

**Proposition 6** *If  $(S, d_1), (S, d_2)$  are metric spaces, and if  $d_1$  and  $d_2$  are equivalent, then a sequence  $\{x_n\}$  converges to  $x_0$  in  $(S, d_1)$  iff it converges to  $x_0$  in  $(S, d_2)$*

**Proof.** Let's assume that  $\{x_n\}$  converges to  $x_0$  in  $(S, d_1)$ , and let's prove that  $\{x_n\}$  converges to  $x_0$  in  $(S, d_2)$ .

Fix  $\epsilon > 0$ . We need to find  $N \in \mathbb{N}$ , such that for all  $n > N$ ,  $d_2(x_n, x_0) < \epsilon$ .

By the equivalence of  $d_1$  and  $d_2$ , we know that there exists  $C > 0$  such that  $d_2(x_n, x_0) \leq C d_1(x_n, x_0)$  (for all  $n$ ).

By the convergence of  $\{x_n\}$  to  $x_0$  in  $(S, d_1)$ , we know that there exists  $N_1 \in \mathbb{N}$ , such that for all  $n > N_1$ ,  $d_1(x_n, x_0) < \frac{\epsilon}{C}$ .

Then, by choosing  $N = N_1$  we know that, for all  $n > N$ ,

$$\begin{aligned} d_2(x_n, x_0) &\leq C d_1(x_n, x_0) \\ &< C \frac{\epsilon}{C} \\ &= \epsilon \end{aligned}$$

■

We now introduce the notion of a bounded sequence. Basically, we will say that a sequence is bounded if all of its terms are not farther from an arbitrary point  $y$  than a constant  $K$ .

**Definition 7** A sequence  $\{x_n\}$  in  $(S, d)$  is bounded iff there exist a point  $y \in S$  and a constant  $K \in \mathbb{R}$  such that  $d(x_n, y) \leq K$  for all  $n \in \mathbb{N}$ .

Being bounded is in some sense a weak requirement, since it's necessary, but not sufficient, for convergence.

**Proposition 8** If a sequence  $\{x_n\}$  in  $(S, d)$  converges to  $x_0$ , then it is bounded.

**Proof.** Since  $x_n \rightarrow x_0$ , there exists  $N \in \mathbb{N}$ , such that for all  $n > N$ ,  $d(x_n, x_0) \leq 1$ . From this we know that the terms with  $n > N$  are close to  $x_0$  (distance less than 1). For the rest of the terms, since there is only a finite number of them, we consider their maximum distance to  $x_0$ , so we define  $K_0 = \max\{d(x_1, x_0), \dots, d(x_N, x_0)\}$ . With this we have, for all  $n \in \mathbb{N}$ ,

$$d(x_n, x_0) \leq \max\{K_0, 1\} \equiv K.$$

■

**Remark 9** The boundedness of a sequence is far from sufficient for convergence, as shown by the sequence defined by  $x_n = (-1)^n$ . It is obviously bounded, since  $d(x_n, 0) = 1$ , and from the example 4, we know that it does not converge.

**Remark 10** If a sequence is bounded, it is easy to see that, for any  $y'$  there exists a  $K'$  such that  $d(x_n, y') \leq K'$ . The proof is left to the reader.

The next definition introduces the notion of Cauchy sequences. A sequence is Cauchy if its elements get close to each other up to any  $\epsilon$ , provided that their indices are big enough.

**Definition 11** A sequence  $\{x_n\}$  in  $(S, d)$  has the Cauchy property iff for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $m, n > N$ ,  $d(x_m, x_n) < \epsilon$

If a sequence converges, its elements get closer and closer to the limit and, in particular, they get close to each other, thus satisfying the Cauchy property. This is proved in the next proposition.

**Proposition 12** If a sequence  $\{x_n\}$  in  $(S, d)$  converges to some  $x_0$ , then it has the Cauchy property.

**Proof.** Fix  $\epsilon > 0$ . Since  $x_n \rightarrow x_0$ , there exists  $N \in \mathbb{N}$ , such that for all  $n > N$ ,  $d(x_n, x_0) \leq \frac{\epsilon}{2}$ . Then, for all  $m, n > N$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_0) + d(x_0, x_m) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

■

The other implication is not always true. There exist metric spaces  $(S, d)$  such that Cauchy sequences do not converge, as the next example shows. Nonetheless, as we will see later, if  $S = \mathbb{R}$  or  $S = \mathbb{R}^n$  with the usual distances, then Cauchy sequences converge (see proposition ??).

**Example 13** Consider  $S = \mathbb{Q}$  and  $d(x, y) = |x - y|$ , Then the sequence defined by

$$\begin{aligned}x_1 &= 3.1 \\x_2 &= 3.14 \\&\dots \\x_n &= \pi \text{ truncated to the } n\text{th decimal}\end{aligned}$$

obviously has the Cauchy property, since for all  $m, n > N$ ,  $|x_n - x_m| < 10^{-N}$ , but it does not converge to an element of  $\mathbb{Q}$ , since  $x_n \rightarrow \pi \notin \mathbb{Q}$ .