Sequences in $\mathbb{R}$

We know examine some special properties of sequences when $S = \mathbb{R}$. Most of them will be easy generalized to $\mathbb{R}^n$, as we will see later. We start with a proposition that simplifies the calculus of limits a lot.

Proposition 1 *Consider two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{R}$ such that $x_n \longrightarrow x_0$ and $y_n \longrightarrow y_0$. Then*
(i)$x_n + y_n \longrightarrow x_0 + y_0$
(ii)$x_n - y_n \longrightarrow x_0 - y_0$
(iii)$x_n y_n \longrightarrow x_0 y_0$
(iv)If $y_0 \neq 0$, then $\frac{x_n}{y_n} \longrightarrow \frac{x_0}{y_0}$

Proof.
We will prove just (i) and (iii), and leave (ii) and (iv) as an exercise.
(i)Fix $\epsilon > 0$. We need to find N such that for all $n > N$, $|(x_n+y_n)-(x_0+y_0)| < \epsilon$.
By the convergence of $\{x_n\}$ and $\{y_n\}$, we have that
there exists $N_1 \in \mathbb{N}$, such that for all $n > N_1$, $|x_n - x_0| < \frac{\epsilon}{2}$
and there exists $N_2 \in \mathbb{N}$, such that for all $n > N_2$, $|y_n - y_0| < \frac{\epsilon}{2}$
Then, choosing $N > \max\{N_1, N_2\}$, we have that, for all $n > N$

$$\begin{aligned} |(x_n + y_n) - (x_0 + y_0)| &\leq |x_n - x_0| + |y_n - y_0| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

(iii)Fix $\epsilon > 0$. We need to find N such that for all $n > N$, $|x_n y_n - x_0 y_0| < \epsilon$.
Since $\{x_n\}$ converges, we know that is bounded, so there exists K s.t. $|x_n| < K$.
By the convergence of $\{x_n\}$ and $\{y_n\}$, we have that
there exists $N_1 \in \mathbb{N}$, such that for all $n > N_1$, $|x_n - x_0| < \frac{\epsilon}{2|y_0|}$. (If $y_0 = 0$, then it is enough to make $|x_n - x_0| < C$, with C any positive real number)
and there exists $N_2 \in \mathbb{N}$, such that for all $n > N_2$, $|y_n - y_0| < \frac{\epsilon}{2K}$
Then, choosing $N > \max\{N_1, N_2\}$, we have that, for all $n > N$

$$\begin{aligned} |x_n y_n - x_0 y_0| &\leq |x_n y_n - x_n y_0| + |x_n y_0 - x_0 y_0| \\ &= |x_n||y_n - y_0| + |y_0||x_n - x_0| \\ &< K\frac{\epsilon}{2K} + |y_0|\frac{\epsilon}{2|y_0|} \\ &= \epsilon \end{aligned}$$

■

In $\mathbb{R}$ (as in $\mathbb{R}^n$) Cauchy sequences (see definition **??**)converge. This is a particularly important result since the definition of Cauchy sequences does not involve a priori knowledge of the limit, therefore the next proposition allows us to prove that a sequence converges without knowing the limit.

Proposition 2 *A sequence $\{x_n\}$ in $\mathbb{R}$ has the Cauchy property iff it converges.*

Proof. To be added ■