Constrained Optimization

We start first with the problem of equality constraints.

Theorem 1 *Suppose that there is an objective function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a collection of functions that define constraints $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., m$. We want to solve the following problem:*

$$\begin{array}{cl} \max\limits_{x \in \mathbb{R}^n} & f(x) \\ s.t. & g_1(x) = 0 \\ & g_2(x) = 0 \\ & ... \\ & g_m(x) = 0 \end{array}$$

If x^* is a solution to the problem, then there exists numbers $\lambda_1, ..., \lambda_m$ such that

$$\frac{\partial f(x^*)}{\partial x_i} = \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x^*)}{\partial x_i} \text{ for all } i = 1, ..., n \tag{1}$$

Remark 2 *The previous theorem gives us a way to look for local maxima. We need to find $x_1^*, ..., x_n^*$, that together with $\lambda_1, ..., \lambda_m$ (also called the Lagrange multipliers) satisfy the conditions in* (1)*. These n conditions, plus the m conditions from feasibility $g_i(x^*) = 0$ give us $m + n$ equations to find the $m + n$ unknowns (n variables x_i and m multipliers λ_j).*

Example 3 *1.Consider a typical problem in economics, a consumer that maximizes utility subject to a budget constraint, for example*

$$\begin{array}{cl} \max\limits_{x_1, x_2, x_3} & \alpha ln(x_1) + \beta ln(x_2) + \gamma ln(x_3) \\ s.t. & p_1 x_1 + p_2 x_2 + p_3 x_3 \leq W \end{array}$$

Since the utility function $\alpha ln(x_1) + \beta ln(x_2) + \gamma ln(x_3)$ is strictly increasing in each variable, we know that any solution (x_1^*, x_2^*, x_3^*) will have $p_1 x_1^* + p_2 x_2^* + p_3 x_3^* = W$, so we can replace the previous problem by

$$\begin{array}{cl} \max\limits_{x_1, x_2, x_3} & \alpha ln(x_1) + \beta ln(x_2) + \gamma ln(x_3) \\ s.t. & p_1 x_1 + p_2 x_2 + p_3 x_3 = W \end{array}$$

This correspond exactly to the framework of our theorem, with $n = 3$, $m = 1$, $f(x_1, x_2, x_3) = \alpha ln(x_1) + \beta ln(x_2) + \gamma ln(x_3)$ and $g_1(x_1, x_2, x_3) = p_1 x_1 + p_2 x_2 + p_3 x_3 - W$.

The conditions in (1) *then become*

$$\begin{aligned} \frac{\partial f(x^*)}{\partial x_1} = \lambda_1 \frac{\partial g(x^*)}{\partial x_1} \quad &\Longrightarrow \quad \frac{\alpha}{x_1^*} = \lambda_1 p_1 \\ \frac{\partial f(x^*)}{\partial x_2} = \lambda_1 \frac{\partial g(x^*)}{\partial x_2} \quad &\Longrightarrow \quad \frac{\beta}{x_2^*} = \lambda_1 p_2 \\ \frac{\partial f(x^*)}{\partial x_3} = \lambda_1 \frac{\partial g(x^*)}{\partial x_3} \quad &\Longrightarrow \quad \frac{\gamma}{x_3^*} = \lambda_1 p_3 \end{aligned}$$

We get then

$$p_1 x_1^* = \frac{\alpha}{\lambda_1} \tag{2}$$

$$p_2 x_2^* = \frac{\beta}{\lambda_1} \tag{3}$$

$$p_3 x_3^* = \frac{\gamma}{\lambda_1} \tag{4}$$

The fourth equation comes from feasibility, so we have

$$p_1 x_1^* + p_2 x_2^* + p_3 x_3^* = W \tag{5}$$

Substituting $(6) - (8)$ *into* (9) *we get* $\frac{\alpha}{\lambda_1} + \frac{\beta}{\lambda_1} + \frac{\gamma}{\lambda_1} = W$*, which gives* $\lambda_1 = \frac{\alpha+\beta+\gamma}{W}$*. Substituting this last expression back into* $(6) - (8)$ *we get*

$$\begin{aligned} x_1^* &= \frac{\alpha W}{p_1(\alpha+\beta+\gamma)} \\ x_2^* &= \frac{\beta W}{p_2(\alpha+\beta+\gamma)} \\ x_3^* &= \frac{\gamma W}{p_3(\alpha+\beta+\gamma)} \end{aligned}$$

Now we provide numerical examples of the problems covered above.

Example 4 *1.Consider a typical problem in economics, a consumer that maximizes utility subject to a budget constraint, for example*

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & 2ln(x_1) + ln(x_2) + 5ln(x_3) \\ s.t. \quad & x_1 + 2x_2 + 4x_3 \leq 1 \end{aligned}$$

Since the utility function $2ln(x_1) + ln(x_2) + 5ln(x_3)$ *is strictly increasing in each variable, we know that any solution* (x_1^*, x_2^*, x_3^*) *will have* $x_1^* + 2x_2^* + 4x_3^* = W$*, so we can replace the previous problem by*

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & 2ln(x_1) + ln(x_2) + 5ln(x_3) \\ s.t. \quad & x_1 + 2x_2 + 4x_3 = 1 \end{aligned}$$

This correspond exactly to the framework of our theorem, with $n = 3$, $m = 1$, $f(x_1, x_2, x_3) = 2ln(x_1)+ln(x_2)+5ln(x_3)$ *and* $g_1(x_1, x_2, x_3) = x_1+2x_2+4x_3-1$.

The conditions in (1) *then become*

$$\begin{array}{rcl}
\frac{\partial f(x^*)}{\partial x_1} = \lambda_1 \frac{\partial g(x^*)}{\partial x_1} & \Longrightarrow & \frac{2}{x_1^*} = \lambda_1 \\
\frac{\partial f(x^*)}{\partial x_2} = \lambda_1 \frac{\partial g(x^*)}{\partial x_2} & \Longrightarrow & \frac{1}{x_2^*} = 2\lambda_1 \\
\frac{\partial f(x^*)}{\partial x_3} = \lambda_1 \frac{\partial g(x^*)}{\partial x_3} & \Longrightarrow & \frac{5}{x_3^*} = 4\lambda_1
\end{array}$$

We get then

$$x_1^* = \frac{2}{\lambda_1} \tag{6}$$

$$2x_2^* = \frac{1}{\lambda_1} \tag{7}$$

$$4x_3^* = \frac{5}{\lambda_1} \tag{8}$$

The fourth equation comes from feasibility, so we have

$$x_1^* + 2x_2^* + 4x_3^* = W \tag{9}$$

Substituting $(6) - (8)$ *into* (9) *we get* $\frac{2}{\lambda_1} + \frac{1}{\lambda_1} + \frac{5}{\lambda_1} = 1$, *which gives* $\lambda_1 = 8$. *Substituting this last expression back into* $(6) - (8)$ *we get*

$$\begin{array}{rcl}
x_1^* & = & \frac{2}{8} \\
x_2^* & = & \frac{1}{16} \\
x_3^* & = & \frac{5}{32}
\end{array}$$

For the analysis in the previous theorem to be correct, there is one more condition that has to be satisfied, usually referred to in the literature as the **qualification constraint**. This condition is in general satisfied in the typical problems in economics, as we will show in the examples.

Definition 5 *The qualification constraint is satisfied at the "candidate point"* x^* *iff the matrix*

$$\begin{bmatrix} \frac{\partial g_1(x^*)}{\partial x_1} & \cdots & \frac{\partial g_1(x^*)}{\partial x_N} \\ \cdots & & \cdots \\ \frac{\partial g_1(x^*)}{\partial x_1} & \cdots & \frac{\partial g_1(x^*)}{\partial x_N} \end{bmatrix}$$

is of rank M.

Notice that the matrix has M rows (one for each constraint) and N columns (one for each variable). Then the only interesting case is when $M < N$, so the variables are not completely determined by the constraint and it is really possible to maximize.

Example 6 *1.Consider any problem of consumer maximization, that is*

$$\begin{array}{cl} \max\limits_{x_i} & u(x_1, x_2, ..., x_n) \\ s.t. & p_1x_1 + p_2x_2 + ... + p_nx_n = W \end{array}$$

Then, M=1 and the condition to check is that the matrix

$$[p_1p_2...p_n]$$

is of rank 1. That is equivalent to check that the matrix does not have only 0s. Since the price vector is usually strictly positive (if not, the consumer would demand an infinite amount of some goods), the qualification constraint is satisfied.

2.Consider two agents with utility $u_i(x, y) = \alpha ln(x) + \beta ln(y)$. The total endowment in the society of good x is $\bar{x}$ and of good y is $\bar{y}$. The problem of finding the Pareto Optimal allocations in this economy can be formulated as

$$\begin{array}{cl} \max\limits_{x_1,y_1,x_2,y_2} & \alpha ln(x_1) + \beta ln(y_1) \\ s.t. & \alpha ln(x_2) + \beta ln(y_2) \geq \bar{u} \\ & x_1 + x_2 = \bar{x} \\ & y_1 + y_2 = \bar{y} \end{array}$$

Since the objective function is strictly increasing in each variable, it is easy to see that at any maximum, the first constraint will be satisfied with equality, so we can write

$$\begin{array}{cl} \max\limits_{x_1,x_2,y_1,y_2} & \alpha ln(x_1) + \beta ln(y_1) \\ s.t. & \alpha ln(x_2) + \beta ln(y_2) = \bar{u} \\ & x_1 + x_2 = \bar{x} \\ & y_1 + y_2 = \bar{y} \end{array}$$

In this case, $M = 3$ and $N = 4$ (the variables are x_1, y_1, x_2, y_2). The condition to check is that at any candidate point $(x_1^*, y_1^*, x_2^*, y_2^*)$ the matrix

$$\begin{bmatrix} 0 & \frac{\alpha}{x_2^*} & 0 & \frac{\beta}{y_2^*} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

has rank 3. This is verified since for any x_2^*, y_2^*, *the quantities* $\frac{\alpha}{x_2^*}$ *and* $\frac{\beta}{y_2^*}$ *are strictly positive.*

We now provide numerical examples

Example 7 *1.Consider a problem of consumer maximization, for example*

$$\begin{array}{cl} \max\limits_{x_i} & ln(x_1) + 3ln(x_2) + ln(x_3) \\ s.t. & 2x_1 + 5x_2 + x_3 = 87 \end{array}$$

Then, M=1 and the condition to check is that the matrix

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is of rank 1, which is trivially satisfied.

2.Consider two agents with utility $u_i(x, y) = 9ln(x) + 10ln(y)$. *The total endowment in the society of good* x *is* 78 *and of good* y *is* 86. *The problem of finding the Pareto Optimal allocations in this economy can be formulated as*

$$\begin{array}{cl} \max\limits_{x_1,y_1,x_2,y_2} & \alpha 9ln(x_1) + \beta 10ln(y_1) \\ s.t. & \alpha 9ln(x_2) + \beta 10ln(y_2) \geq \bar{u} \\ & x_1 + x_2 = \bar{78} \\ & y_1 + y_2 = \bar{86} \end{array}$$

Since the objective function is strictly increasing in each variable, it is easy to see that at any maximum, the first constraint will be satisfied with equality, so we can write

$$\begin{array}{cl} \max\limits_{x_1,x_2,y_1,y_2} & 9ln(x_1) + 10ln(y_1) \\ s.t. & 9ln(x_2) + 10ln(y_2) = \bar{u} \\ & x_1 + x_2 = 78 \\ & y_1 + y_2 = 86 \end{array}$$

In this case, $M = 3$ *and* $N = 4$ *(the variables are* x_1, y_1, x_2, y_2*). The condition to check is that at any candidate point* $(x_1^*, y_1^*, x_2^*, y_2^*)$ *the matrix*

$$\begin{bmatrix} 0 & \frac{9}{x_2^*} & 0 & \frac{10}{y_2^*} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

has rank 3. This is verified since for any x_2^*, y_2^*, *the quantities* $\frac{9}{x_2^*}$ *and* $\frac{10}{y_2^*}$ *are strictly positive.*