A Review of Derivatives

We will make a quick review of the rules of derivation of real-valued functions. The first set of results and examples corresponds to functions $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Definition 1 *The derivative of the function f at the point x is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Theorem 2 *(Basic results)*

$$\begin{array}{ll} f(x) = k & f'(x) = 0 \\ f(x) = ax & f'(x) = a \\ f(x) = x^n & f'(x) = nx^{n-1} \\ f(x) = ln(x) & f'(x) = \frac{1}{x} \\ f(x) = e^x & f'(x) = e^x \end{array}$$

Example 3 *1.$h(x) = \sqrt{x}$.*
Noticing that $\sqrt{x} = x^{\frac{1}{2}}$, and applying the third result in the previous list we get

$$\begin{aligned} h'(x) &= \frac{1}{2}x^{\frac{1}{2}-1} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Theorem 4 *(Basic operations)*

$$\begin{aligned} (f(x) + g(x))' &= f'(x) + g'(x) \\ (f(x)g(x))' &= f'(x)g(x) + f(x)g'(x) \\ \left(\frac{f(x)}{g(x)}\right)' &= \frac{f'(x)g(x) - f(x)g'(x))}{(g(x))^2} \end{aligned}$$

Example 5 *1.$h(x) = \frac{1}{x}$.*
Using the third result in theorem 4, with $f(x) = 1$ and $g(x) = x$, we get

$$\begin{aligned} h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{0 \cdot 1 - 1 \cdot 1}{x^2} \\ &= -\frac{1}{x^2} \end{aligned}$$

2.$h(x) = xe^x$.
Using the second result in theorem 4, with $f(x) = x$ and $g(x) = e^x$, we get

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 1 \cdot e^x - xe^x \\ &= (1 - x)e^x \end{aligned}$$

3. $h(x) = a \cdot g(x)$.
Using the second result in theorem 4, with $f(x) = a$ *we get*

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 0 \cdot g(x) + a \cdot g'(x) \\ &= a \cdot g'(x) \end{aligned}$$

4. $h(x) = ax^2 + bx + c$.
Using the first result in theorem 4 and the previous exercise we get

$$\begin{aligned} h'(x) &= (ax^2)' + (bx)' + (c)' \\ &= a \cdot (x^2)' + b \cdot (x)' + 0 \\ &= 2ax + b \end{aligned}$$

Theorem 6 *(Chain Rule) Consider* $h(x) = f \circ g(x)$. *Then* $h'(x) = f'(g(x))g'(x)$.

Example 7 *1.Consider* $h(x) = e^{x^2}$. *Then* $h(x) = f \circ g(x)$ *with* $f(x) = e^x$ *and* $g(x) = x^2$. *Using theorem 6, and remembering that in this case* $f'(x) = e^x$ *and* $g'(x) = 2x$ *we get*

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= e^{x^2} \cdot 2x \end{aligned}$$

2.Consider $h(x) = ln(x^p)$. *Then* $h(x) = f \circ g(x)$ *with* $f(x) = ln(x)$ *and* $g(x) = x^p$. *Using theorem 6, and remembering that in this case* $f'(x) = \frac{1}{x}$ *and* $g'(x) = px^{p-1}$ *we get*

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{x^p} px^{p-1} \\ &= \frac{p}{x} \end{aligned}$$

(Notice that the same result could be found by noticing that $ln(x^p) = pln(x)$ *and taking derivative of this function).*

We now consider functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and introduce the concept of **partial derivative**. Basically, it corresponds to "fix" all the variables except one (consider them constants) and then take derivative with respect to the other one. The technical definition is the following

Definition 8 $\frac{\partial f(x_1,\ldots,x_n)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1,\ldots,x_{i-1},x_i+h,x_{i+1},\ldots x_n) - f(x_1,\ldots,x_n)}{h}$

We now illustrate the concept with some examples of utility functions that often appear in economic problems.

Example 9 *1.*$f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.
To calculate $\frac{\partial f(x_1,x_2)}{\partial x_1}$ *we consider* x_2 *as a constant, then we get:*

$$\begin{aligned}\frac{\partial f(x_1, x_2)}{\partial x_1} &= x_2^{1-\alpha}(x_1^{\alpha})' \\ &= x_2^{1-\alpha}(\alpha x_1^{\alpha-1}) \\ &= \alpha x_1^{\alpha-1} x_2^{1-\alpha}\end{aligned}$$

Analogously, one could get

$$\begin{aligned}\frac{\partial f(x_1, x_2)}{\partial x_2} &= x_1^{\alpha}(x_2^{1-\alpha})' \\ &= x_1^{\alpha}((1-\alpha)x_2^{-\alpha}) \\ &= (1-\alpha)x_1^{\alpha} x_2^{-\alpha}\end{aligned}$$

2.$f(x_1, x_2) = (x_1^{\rho} + x_2^{\rho})^{\frac{1}{\rho}}$.
Again, considering x_2 *as a constant, and using the chain rule we get*

$$\begin{aligned}\frac{\partial f(x_1, x_2)}{\partial x_1} &= \frac{1}{\rho}(x_1^{\rho} + x_2^{\rho})^{\frac{1}{\rho}-1} \cdot (\rho x_1^{\rho-1}) \\ &= (x_1^{\rho} + x_2^{\rho})^{\frac{1}{\rho}-1} \cdot x_1^{\rho-1}\end{aligned}$$