Optimization Without Constraints

This corresponds to the case of $K = \mathbb{R}^n$. We first state the basic first order condition, which is necessary (but not sufficient) for x^* to be a local maximum.

Theorem 1 *If x^* is a local maximum of $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, then $\nabla f(x^*) \equiv (\frac{\partial f(x^*)}{\partial x_1}, ..., \frac{\partial f(x^*)}{\partial x_n}) = (0, ..., 0)$.*

Corollary 2 *If we consider a single-dimensional function, that is $f : \mathbb{R} \longrightarrow \mathbb{R}$, and x^* is a local maximum of f, then $f'(x^*) = 0$.*

What conditions can assure us that a point x^* is indeed a local maximum? We state a useful condition in the next result.

Theorem 3 *If $\nabla f(x^*) = 0$ and $Hf(x^*) = \begin{bmatrix} \frac{\partial^2 f(x^*)}{\partial x_1 \partial x_1} & ... & \frac{\partial^2 f(x^*)}{\partial x_1 \partial x_n} \\ & ... & \\ \frac{\partial^2 f(x^*)}{\partial x_n \partial x_1} & ... & \frac{\partial^2 f(x^*)}{\partial x_n \partial x_n} \end{bmatrix}$ is negative definite, then x^* is a local maximum of f.*

Corollary 4 *If we consider a single-dimensional function, that is $f : \mathbb{R} \longrightarrow \mathbb{R}$, and $f'(x^*) = 0$ and $f''(x^*) < 0$, then x^* is a local maximum.*

Finally, we address the problem of finding conditions for a global maximum. This are in general difficult to satisfy, but there is one which is very useful in economics.

Theorem 5 *If f is strictly concave ($Hf(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_1} & ... & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ & ... & \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & ... & \frac{\partial^2 f(x)}{\partial x_n \partial x_n} \end{bmatrix}$ for all $x \in \mathbb{R}^n$) and $f'(x^*) = 0$, then x^* is a global maximum. Moreover, if such an x^* exists, then it is unique.*

Example 6 *1.$f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = ax^2 + bx + c$, $a < 0$.*
In this case, the corollary applies, and we just need to find x^* such that $f'(x^*) = 0$. That is, $2ax^* + b = 0$, which finally gives us $x^* = -\frac{b}{2a}$. How do we know that x^* is indeed a maximum? By looking at corollary 4 we just need to check that $f''(x^*) < 0$, which is true since $f''(x^*) = a < 0$.

2.$f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = ax^2 + bx + c$, $a > 0$. Doing exactly the same as before we get $x^* = -\frac{b}{2a}$.
But is this really a maximum? We can't be sure, since now $f''(x^*) = a > 0$.
Consider an example, let's say $a = b = c = 1$. Then $x^* = -\frac{1}{2}$ and $f(x^*) = \frac{3}{4}$, but if we consider any other $\bar{x}$, like $\bar{x} = 0$, we get $f(\bar{x}) = 1 > f(x^*)$. What is wrong? Well, in fact x^* is not a maximum, it is a minimum!
This shows that the condition $f'(x^*) = 0$, is necessary but not sufficient: the fact that a point satisfies this condition makes it a candidate for a maximum, it

does not assure us that it is so.

3.Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = -x^4 + 2x^3 + 2x^2 + 10$. *The points that satisfy* $f'(x^*) = -4x^3 + 6x^2 + 4x = 0$ *are* $x_1^* = -\frac{1}{2}$, $x_2^* = 0$ *and* $x_3^* = 2$. *Which ones are local maxima? For that we need to compute* $f''(x) = -12x^2 + 12x + 4$ *and evaluate it at* x_i^*. *We get*

$$\begin{aligned} f''(x_1^*) &= -5 < 0 \\ f''(x_2^*) &= 4 > 0 \\ f''(x_3^*) &= -20 < 0 \end{aligned}$$

Then, the two points that are local maxima are $x_1^* = -\frac{1}{2}$ *and* $x_3^* = 2$. *Is any of them a global maximum? To answer that, notice first that the function goes to* $-\infty$ *when* x *goes to plus or minus infinity, so the local maxima are candidate for global maxima. Noticing that* $f(-\frac{1}{2}) = 10$ *and* $f(2) = 18$, *we conclude that 2 is a global maximum.*

4.Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = ax + b$. *Is there a local a maximum? It is easy to see that* $f'(x) = a$ *for all* x. *Then, if* $a \neq 0$, *the function does not have a local maximum, since there is no point that satisfies the necessary condition. This coincides with the natural result: if* $a > 0$ *then the function is strictly increasing, and for every point* x^*, *a point to the right of it gives a higher value of the objective function. Analogously, if* $a < 0$ *the function is strictly decreasing, and for every point* x^*, *a point to the left of it gives a higher value of the objective function.*

5.Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *given by* $f(x, y) = xye^{-x-y}$.
The condition $\nabla f(x^*, y^*) = 0$ *gives us*

$$\begin{aligned} \frac{\partial f(x^*, y^*)}{\partial x} &= y^* e^{-y^*}(e^{-x^*} - x^* e^{-x^*}) = 0 \\ \frac{\partial f(x^*, y^*)}{\partial y} &= x^* e^{-x^*}(e^{-y^*} - y^* e^{-y^*}) = 0 \end{aligned}$$

We get two candidates for a local maximum: $(x_1^*, y_1^*) = (0, 0)$ *and* $(x_2^*, y_2^*) = (1, 1)$. *To check if one or both are indeed local maxima, we need to find* $Hf(0, 0)$ *and* $Hf(1, 1)$. *A little bit of computation gives us:*

$$Hf(x, y) = \begin{bmatrix} ye^{-y}((x-2)e^{-x}) & (1-x)e^{-x}(1-y)e^{-y} \\ (1-x)e^{-x}(1-y)e^{-y} & xe^{-x}((y-2)e^{-y}) \end{bmatrix}$$

Evaluating we get $Hf(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and* $Hf(1, 1) = \begin{bmatrix} -e^{-2} & 0 \\ 0 & -e^{-2} \end{bmatrix}$. *Only the second matrix is negative definite, so we can only be sure that* $(1, 1)$ *is a local maximum.*