Random Variables

In many experiments it is easier to deal with a summary variable than with the original probability structure. Suppose than you want to flip a coin 100 times. The sample space will contain 2^{100} elements, with each element being an ordered string of heads and tails. Random variables are tools that transform the sample space into some other space that is more tractable. Define the random variable X as the number of heads obtained in the 100 tosses. Thus, the sample space of X is the set of integers $\{1, 2, \cdots, 100\}$ which is much easier and cleaner to deal with, than the original sample space.

A **random variable** X: $\Omega \rightarrow \Re$ is a function from the sample space Ω to the reals.

In defining a random variable we indicate that the original sample space is mapped into another space (usually a set of reals). Suppose that initially, we are dealing with the sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ with a probability function P and we define a random variable X with range $X = \{x_1, x_2, \cdots, x_n\}$. We can define a probability function P_X on X in the following way: we will observe $X = x_i$ if and only if the outcome of the random experiment is an $\omega_j \in \Omega$.

Thus, $P_X(X = x_i) = P(\{\omega_j \in \Omega : X(\omega_j) = x_i\})$. P_X formally defines a probability function induced by the random variable X.

A note on notation: Random variables will always be denoted with uppercase letters and the realized values of the random variable (its range) will be denoted by the corresponding lowercase letters.

Example

Consider again the tossing of a fair coin 3 times. Define the random variable X to be the number of heads obtained in the three tosses. A complete enumeration of the value of X for each point in the original sample space is:

ω	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
X(ω)	3	2	2	2	1	1	1	0

Therefore, the range of the random variable X is $X = \{0, 1, 2, 3\}$. Since the coin is fair each sample point ω has a probability of $\frac{1}{8}$.

By simply counting the above display we can derive the induced probability function on X.

x	0	1	2	3
P_X(X= x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

For example, $P_X(X=1) = P(\{HTT, TTH, THT\}) = \frac{3}{8}$.