A factorization of a matrix $A$ is an equation that expresses $A$ as a product of two or more matrices.

**The LU Decomposition**

The LU decomposition, is a procedure where a matrix $A$ of $m \times n$ dimension, is decomposed into a *lower triangular matrix* $L$ (with 1’s on the diagonal) of dimension $m \times m$, and an *upper triangular matrix* $U$ of dimension $m \times n$. Thus, $A$ is factorized into $L$ and $U$. Such a decomposition is called the **LU decomposition** of $A$.

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} LE & * & * & * \\ 0 & LE & * & * \\ 0 & 0 & 0 & LE \\ 0 & 0 & 0 & 0 & LE \end{bmatrix}$$

Note that the starred (*) entries can be any real numbers, whereas “$LE$” defined as $\mathbb{R}\setminus\{0\}$, stands for the leading entry in the row.
An LU Factorization Algorithm

Suppose $A$ can be reduced to an echelon form $U$ (recall that $U$ is an $m \times n$ echelon form of $A$) without row interchanges. Then since row scaling is not essential, $A$ can be reduced to $U$ with only row replacements (replacement of one row by the sum of itself and a multiple of another row).

Consequently, there exist unit lower triangular elementary matrices $E_1, E_2, \cdots, E_p$ such that $(E_p \cdots E_1)A = U$.

Therefore, $A = IA = (E_p \cdots E_1)^{-1}(E_p \cdots E_1)A = (E_p \cdots E_1)^{-1}U = LU$, if we define $L := (E_p \cdots E_1)^{-1}$.

To summarize:

1. Determine the dimension of $U$ and $L$ based on the dimension of $A$.
2. Reduce $A$ to an echelon form $U$ by a sequence of only row replacement operations, if possible.
3. Place entries in $L$ such that the same sequence of row operations reduces $L$ to $I$.

Example

Find an LU factorization of $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 8 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

Step 1: Determine the dimension of $U$ and $L$ based on the dimension of $A$. Since $A$ is a $3 \times 3$ matrix, $L$ should be $3 \times 3$ and so must $U$.

Step 2: Reduce $A$ to an echelon form $U$ by a sequence of only row replacement operations.

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Then,

$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 2 & 8 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 12 & 3 \\ 0 & -2 & 0 \end{bmatrix}$;
Then,

\[ E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{6} & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 12 & 3 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 12 & 3 \\ 0 & 0 & \frac{1}{7} \end{bmatrix} = U \]

Hence, \( A = L(E_2E_1A) \)

Step 3: Place entries in \( L \) such that the same sequence of row operations reduces \( L \) to \( I \).

\[ L = (E_2E_1)^{-1} = E_1^{-1}E_2^{-1} \]

\[ E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \]

\[ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{6} & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{6} & 1 \end{bmatrix} \]

Therefore,

\[ L = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{6} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{1}{6} & 1 \end{bmatrix} \]

Again it is a good idea to verify that \( L \) and \( U \) satisfy \( LU = A \).

\[ LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{1}{6} & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 12 & 3 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 8 & 1 \\ -1 & 0 & 1 \end{bmatrix} \]