

Lecture 1*

Consider the following description of the world:

- Infinite horizon
- Time is discrete
- There is a finite set of events $S = \{s^1, \dots, s^S\}$, and
- There exists a representative household endowed in each period with stochastic stream of consumption good, $y_t(s^t)$, at each period. Where $s^t = \{s_0, \dots, s_t\}$ denotes the history of events $s_t \in S = \{s^1, \dots, s^S\}$ up to date t . We denote probability of any history s^t by $\mu(s^t)$.
- Household chooses to consume or save and seeks to maximize the following objective function ($0 < \beta < 1$)

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where E_0 denotes expectation at time 0.

Planner's Problem

Planner chooses allocations $c_t(s^t)$ to solve

$$\max_{\{c_t(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t(s^t)) \mu(s^t)$$

$$\text{s.t.} \quad c_t(s^t) \leq y_t(s^t) \quad \forall t, s^t$$

When $u(\cdot)$ is increasing the solution to planner's problem is

$$c_t(s^t) = y_t(s^t) \quad \forall t, s^t$$

Decentralization

An *Arrow-Debreu Equilibrium* is allocations $\{c_t(s^t)\}$ and pricing system $\{q_t(s^t)\}$ for t and all s^t , such that

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i. Given pricing system $\{q_t(s^t)\}$, $\{c_t(s^t)\}$ solve household's problem

$$\begin{aligned} & \max_{\{c_t(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t(s^t)) \mu(s^t) \\ \text{s.t.} \quad & \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) c_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) y_t(s^t) \\ & c_t(s^t) \geq 0 \quad \forall t, s^t \end{aligned}$$

ii. Resource Feasibility

$$c_t(s^t) = y_t(s^t) \quad \forall t, s^t$$

Let's assume that $u(\cdot)$ is twice continuously differentiable and strictly concave, therefore necessary conditions of household's maximization are also sufficient and the solution is unique. Also assume $y_t(s^t) > 0$ and $u(\cdot)$ satisfies INADA conditions so that the solution is interior. Let λ be lagrange multiplier on household's budget constraint. Then the first order conditions imply

$$\beta^t \mu(s^t) u'(c_t(s^t)) = \lambda q_t(s^t)$$

and decentralization problem has the following equilibrium prices (subject to normalization $q_0(s_0) = 1$)

$$q_t(s^t) = \beta^t \frac{u'(y_t(s^t))}{u'(y_0(s_0))} \mu(s^t)$$

Heterogeneous Agents

Consider now an economy with two types of agents, type 1 and type 2. Agent type 1 is endowed with $y^1(s^t)$ units of consumption good and type 2 is endowed with $y^2(s^t)$ units of consumption good in each period t and each history s^t .

An *Arrow-Debreu Equilibrium* is allocations $\{c_t^i(s^t)\}$ for each type $i = 1, 2$ and pricing system $\{q_t(s^t)\}$ for t and all s^t , such that

i. Given pricing system $\{q_t(s^t)\}$, $\{c_t^i(s^t)\}$ solve household's problem (HH)

$$\begin{aligned} & \max_{\{c_t^i(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t^i(s^t)) \mu(s^t) \quad i = 1, 2 \\ \text{s.t.} \quad & \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) y_t^i(s^t) \\ & c_t^i(s^t) \geq 0 \quad \forall t, s^t \end{aligned}$$

ii. Resource Feasibility (RC)

$$c_t^1(s^t) + c_t^2(s^t) = y_t^1(s^t) + y_t^2(s^t) \quad \forall t, s^t$$

Pareto Optimal Allocations

An allocation $\{c_t^1(s^t), c_t^2(s^t)\}$ is *Pareto Optimal* if

i. is feasible

$$c_t^1(s^t) + c_t^2(s^t) = y_t^1(s^t) + y_t^2(s^t) \quad \forall t, s^t$$

ii. there exists no other feasible allocation $\{\hat{c}_t^1(s^t), \hat{c}_t^2(s^t)\}$ such that

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(\hat{c}_t^i(s^t)) \mu(s^t) \geq \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t^i(s^t)) \mu(s^t)$$

with strict inequality for at least one $i = 1, 2$

Question Let the allocation $\{c_t^1(s^t), c_t^2(s^t)\}$ be solution to decentralization problem. Is this allocation Pareto Optimal?

Answer The answer is yes. But in order for standard proof to work we require the assumption that household budget constraint be compact (so that the household problem be well-defined). To insure that it is required that total value of household endowment under equilibrium pricing system be bounded, i.e.

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) y_t^i(s^t) < \infty$$

If we assume that $y^i(s^t)$ is bounded, this condition comes down to

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) < \infty$$

Exercise Show that the above condition is satisfied for our decentralization problem. State the assumptions that is required on household preferences.

Consider again the heterogeneous agent problem, assume

- $u(\cdot)$ is twice continuously differentiable, strictly concave, satisfies INADA conditions
- $0 < y^i(s^t) < \infty$
- λ^i is lagrange multiplier on agent i 's budget constraint

The first order conditions imply

$$\beta^t \mu(s^t) u'(c_t^i(s^t)) = \lambda^i q_t(s^t)$$

therefore

$$\frac{u'(c_t^1(s^t))}{u'(c_t^2(s^t))} = \frac{\lambda^1}{\lambda^2} \quad \text{independent of } t \text{ and } s^t$$

Now assume $u(\cdot)$ is homothetic, in particular let $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. Then

$$\left(\frac{c_t^1(s^t)}{c_t^2(s^t)}\right)^{-\sigma} = \frac{\lambda^1}{\lambda^2}$$

and $\frac{c_t^1(s^t)}{c_t^2(s^t)}$ is independent of t and s^t . This implies that at each date t and after each history s^t allocation to each agent is a constant fraction of total resources. Also from first order conditions it follows that

$$\beta \frac{u'(c_t^1(s^{t+1}))}{u'(c_t^1(s^t))} = \beta \frac{u'(c_t^2(s^{t+1}))}{u'(c_t^2(s^t))}$$

and if $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$

$$\frac{c_t^1(s^{t+1})}{c_t^1(s^t)} = \frac{c_t^2(s^{t+1})}{c_t^2(s^t)} \quad \forall \quad s^t, s^{t+1}$$

Using Second Welfare Theorem to find equilibrium allocations and prices(Due to Mantel-Negishi)

Consider the following planner's problem with pareto weight γ

$$\max_{\{c_t^1(s^t), c_t^2(s^t)\}} \gamma \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t^1(s^t)) \mu(s^t) + (1-\gamma) \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t^2(s^t)) \mu(s^t)$$

$$\text{s.t.} \quad c_t^1(s^t) + c_t^2(s^t) \leq y_t^1(s^t) + y_t^2(s^t) \quad \forall \quad t, s^t$$

Given γ , the above problem has a solution $\{c_t^1(s^t, \gamma), c_t^2(s^t, \gamma)\}$. We want to construct prices that support these allocations in a decentralized economy. Let $q_0(s_0) = 1$ and define

$$q_t(s^t, \gamma) = \beta^t \frac{u'(c_t^i(s^t, \gamma))}{u'(c_0^i(s_0, \gamma))} \mu(s^t)$$

Now in order to have equilibrium, given the above prices, the allocation $\{c_t^1(s^t, \gamma), c_t^2(s^t, \gamma)\}$ has to be affordable

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t, \gamma) (c_t^i(s^t, \gamma) - y_t^i(s^t)) = 0$$

Now our task is to solve the above (system of redundant) equations in one variable γ .

Question Does the above equation(s) necessarily have a solution for γ ?

Answer The answer is yes. The loose idea goes as the following: if gamma is very small social planner gives a little weight to agent of type 1 and therefore type 1's consumption is small. For γ sufficiently small it is always the case that $c_t^1(s^t) < y_t^1(s^t)$ so the left hand side is negative. The opposite is true for γ large; $c_t^1(s^t) > y_t^1(s^t)$. By theorem of maximum $c_t^1(s^t, \gamma)$ is a continuous function of

γ , therefore $q_t(s^t, \gamma)$ is also a continuous function. Thus the left hand side is a continuous function of γ that takes positive and negative values, therefore by the *Intermediate Value Theorem* there exists a γ that makes it equal to zero.

Thus we have shown that under some general assumptions there exists an equilibrium in decentralized economy.