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Lecture 2*

Pricing Assets

Consider the economy with infinitely lived representative households endowed with $y_t(s^t)$ units of consumption goods where s^t is history of some random events $s_t \in \{s^1, \ldots, s^S\}$. The household solves

$$\max_{\{c_t(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t(s^t))\mu(s^t)$$

$$\begin{aligned} \text{s.t.} \qquad \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) c_t(s^t) &\leq \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) y_t(s^t) \\ c_t(s^t) &\geq 0 \qquad \forall \quad t, s^t \end{aligned}$$

and the resource feasibility implies

$$c_t(s^t) = y_t(s^t) \qquad \forall \quad t, s^t$$

Let $\{c_t(s^t)\}$ and $\{q_t(s^t)\}$ be equilibrium allocation and pricing system.

Example 1 Price of *stock* that pays dividend $d(s^t)$ at each history s^t

$$P(d) = \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) d(s^t)$$

Example 2 A *consol* is an asset that pays one unit of consumption good at every date and history, $d^c(s^t) = 1$

$$P(d^c) = \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t)$$

Example 3 An *option* is an asset that pays $d^o = \max\{d(s^t), 0\}$ at history s^t

$$P(d^o) = \sum_{s^t} q_t(s^t) \max\{d(s^t), 0\}$$

Pricing Kernel is the price of one unit of consumption good at date $t+1$ and history s^{t+1} in terms of a unit of consumption good at date t and history s^t

$${}_t q_{t+1}(s^{t+1}) = \frac{q_{t+1}(s^{t+1})}{q_t(s^t)}$$

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Example 4 Value in terms of good at s^t of a stock that pays $d(s^\tau)$ at each history s^τ, $\tau \geq t$

$$P_t(s^t) = \sum_{\tau=t}^{\infty} \sum_{\{s^\tau : s^t \subseteq s^\tau\}} {}_t q_t \tau(s^\tau) d(s^\tau)$$

Sequential Markets

A *Sequential Market Equilibrium* is allocation $\{c(s^t), A(s^t)\}$ and pricing kernel $\{{}_t q_{t+1}(s^{t+1})\}$ such that,

i. Given $\{{}_t q_{t+1}(s^{t+1})\}$, $\{c(s^t), A(s^t)\}$ solve household problem

$$\max_{\{c_t(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t(s^t))\mu(s^t)$$

$$\begin{aligned} \text{s.t.} \quad c_t(s^t) + \sum_{s^{t+1}|_{s^t}} {}_t q_{t+1}(s^{t+1}) A(s^{t+1}) &\leq A(s^t) + y_t(s^t) \qquad \forall \quad t, s^t \\ c_t(s^t) &\geq 0 \\ A(s^t) &\geq -\bar{A} \end{aligned}$$

ii. Market Clearing

$$\begin{aligned} c_t(s^t) &= y_t(s^t) \\ A(s^t) &= 0 \end{aligned}$$

Claim Any Arrow-Debreu Equilibrium allocation is also a Sequential Market Equilibrium allocation and vice versa(again we need to assume $\sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) < \infty$ and $y(s^t) < \infty$)

Outline of the proof The key step of the proof is to show that by appropriately defined prices and asset holdings the budget constraints in two problems are identical. Then the equivalence of problems follow naturally form the fact that utility function reach its maximum at the equilibrium allocations in both problems.

Step 1 Showing that any allocation satisfying Sequential Market budget constraint is also satisfying an Arrow-Debreu budget constraint.
Starting from a sequential budget constrain,

$$c_t(s^t) + \sum_{s^{t+1}|_{s^t}} {}_t q_{t+1}(s^{t+1}) A(s^{t+1}) \leq A(s^t) + y_t(s^t)$$

Let $q_0(s_0) = 1$ and define $q_{t+1}(s^{t+1}) = {}_t q_{t+1}(s^{t+1}) q_t(s^t)$. Therefore, $q_t(s^t) = {}_1 q_2(s^2) \times \cdots \times {}_{t-1} q_t(s^t)$. Multiply each budget constrain by $q_t(s^t)$ and some over all s^t

$$\sum_{s^t} q_t(s^t) c_t(s^t) + \sum_{s^t} q_t(s^{t+1}) A(s^{t+1}) \leq \sum_{s^t} q_t(s^t) A(s^t) + \sum_{s^t} q_t(s^t) y_t(s^t)$$

Then some over t up to some time $T > 0$[1]

$$\sum_{t=0}^{T}\sum_{s^t} q_t(s^t)c_t(s^t) + \sum_{s^{T+1}} q_{T+1}(s^{T+1})A(s^{T+1}) \leq \sum_{t=0}^{T}\sum_{s^t} q_t(s^t)y_t(s^t)$$

Now let $T \to \infty$, since $A(s^{T+1})$ is bounded and $\sum_{t=0}^{\infty}\sum_{s^t} q_t(s^t) < \infty$ (and hence $q_t(s^t) \to 0$) we have

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t(s^t)c_t(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t(s^t)y_t(s^t)$$

Step2 Showing that any Arrow-Debreu budget constraint can be written in sequential form by appropriately define asset holdings and prices.
Start from the following time zero budget constrain,

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t(s^t)c_t(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t(s^t)y_t(s^t)$$

We need to find $A(s^{t+1})$ and $\bar{A}$. Define

$$_tq_{t+1} = \frac{q_t(s^{t+1})}{q_t(s^t)}$$

Also define,

$$A(s^t) = \sum_{\tau=t}^{\infty}\sum_{s^\tau} \frac{q_\tau(s^\tau)}{q_t(s^t)}(c_\tau(s^\tau) - y_\tau(s^\tau))$$

Then we can rewrite this as

$$A(s^t) = c_t(s^t) - y_t(s^t) + \sum_{\tau=t+1}^{\infty}\sum_{s^\tau} {}_tq_{t+1}(s^{t+1})\frac{q_\tau(s^\tau)}{q_{t+1}(s^{t+1})}(c_\tau(s^\tau) - y_\tau(s^\tau))$$

With some manipulations over the order of summations and using the fact that expectation is over al histories that include s^t we arrive at sequential budget constraint.
It is easy to see that that a candidate for $-\bar{A}$ is $\sum_{t=0}^{\infty}\sum_{s^t} q_t(s^t)y_t(s^t)$.[2]

Step 3 The feasibility assumptions are equal in both problems. So an objective function is being maximized over two equivalent constraint sets. Therefore any choice that attains maximum of the objective function in one problem should also attain its maximum on the other problem as well, otherwise there is a contradiction.

Sequence of Markets with two types of agents

A *Sequential Market Equilibrium* is allocation $\{c^i(s^t), A^i(s^t)\}$ for each type $i = 1, 2$ and pricing system $\{_tq_{t+1}(s^{t+1})\}$ such that,

[1] Note that since $q_t(s^t)$ is measurable w.r.t s^t it can go inside the summation

[2] See LS(2001), pages 156-158

i. Given $\{{}_tq_{t+1}(s^{t+1})\}$, $\{c^i(s^t), A^i(s^t)\}$ solve household problem

$$\max_{\{c_t(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t^i(s^t))\mu(s^t)$$

$$\begin{array}{rrcll} \text{s.t.} & c_t^i(s^t) + \displaystyle\sum_{s^{t+1}|_{s^t}} {}_tq_{t+1}(s^{t+1})A^i(s^{t+1}) & \leq & A^i(s^t) + y_t^i(s^t) & \forall \quad t, s^t \\ & c_t^i(s^t) & \geq & 0 & \\ & A^i(s^t) & \geq & \bar{A}^i & \\ & A^i(s_0) & \text{given} & & \end{array}$$

ii. Market Clearing

$$\begin{array}{rcl} c_t^1(s^t) + c_t^2(s^t) & = & y_t^1(s^t) + y_t^2(s^t) \\ A^1(s^t) + A^2(s^t) & = & 0 \end{array}$$

Example Let $S = \{H, L, \emptyset\}$

$$\begin{array}{lll} y^1(H) = 2 & y^1(L) = 0 & y^1(\emptyset) = 1 \\ y^2(H) = 0 & y^2(L) = 2 & y^2(\emptyset) = 1 \end{array}$$

Let s_t follow a *Markov Chain* [3] with transition matrix

$$\begin{array}{c} \\ H \\ L \\ \emptyset \end{array} \begin{array}{c} \begin{array}{ccc} H & L & \emptyset \end{array} \\ \left[\begin{array}{ccc} p & 1-p & 0 \\ 1-p & p & 0 \\ .5 & .5 & 0 \end{array} \right] \end{array}$$

Assume the process starts at $\emptyset$ (note that the process escapes from this state in second period with probability one and never returns to it). Also assume $A^i(s_0) = 0$. Let $\mu(s^t)\lambda^i(s^t)$ be lagrange multiplier on type i's budget constraint. The first order conditions imply

$c^i(s^t): \quad \beta^t\mu(s^t) = \lambda^i(s^t)$
$A^i(s^{t+1}): \quad \mu(s^t)\lambda^i(s^t){}_tq_{t+1}(s^{t+1}) = \mu(s^{t+1})\lambda^i(s^{t+1})$
therefore the prices are

$${}_tq_{t+1}(s^{t+1}) = \beta^t \frac{u'(c^i(s^{t+1}))}{u'(c^i(s^t))} \frac{\mu(s^{t+1})}{\mu(s^t)} = \beta^t \frac{u'(c^i(s^{t+1}))}{u'(c^i(s^t))} \mu(s_{t+1}|s_t)$$

where the last equality is an implication of markov property. Note also that first order conditions imply

$$\frac{u'(c^1(s^t))}{u'(c^1(s_0))} = \frac{u'(c^1(s^t))}{u'(c^1(s_0))} \qquad \text{for all} \quad s^t$$

and therefore

$$\frac{u'(c^2(s^t))}{u'(c^1(s^t))} = \frac{u'(c^2(s_0))}{u'(c^1(s_0))} = \frac{\lambda^2(s_0)}{\lambda^1(s_0)} = \alpha \quad \text{some constant}$$

[3] For brief introduction to Markov Chains read first chapter in LS(2001). For more formal treatment take a look at SLP(1989), chapter 8.

We can use feasibility at date zero and derive the consumption at date zero by solving the following equation

$$\frac{u'(2 - c^1(s_0))}{u'(c^1(s_0))} = \alpha$$

Assume $u(.)$ strictly concave and INADA. The left hand side is an increasing function of $c^(1s_0)$ and therefore there is solution to $c^1(s_0)$. Also the fact that there is no aggregate fluctuation implies that we can use the same equation for every $c^1(s^t)$. This means $c^1(s^t) = c^1(s_0)$ for all s^t. Let's guess the following allocations as equilibrium allocations [4]

$$\begin{array}{ccc} A^1(H) = -1 & A^1(L) = 1 & c^1(s^t) = 1 \\ A^2(H) = 1 & A^2(L) = -1 & c^2(s^2) = 1 \end{array}$$

Question How did we arrive at this guess?

Answer The reason that this guess works is negative correlation of each household's endowment process and symmetric structure of transition matrix.
By construction this allocations are feasible, to verify that it is an equilibrium we need to show that given the prices derived above, they are affordable at each history s^t.

Exercise Verify that the suggested allocations together with the prices derived from first order conditions is a Sequential Market equilibrium.

Question Do prices depend on transitional probabilities?

Answer Yes, knowing that allocations are constant over time, it is easy to see that

$${}_tq_{t+1}(s_{t+1}|s_t) = \beta^t \mu(s_{t+1}|s_t)$$

[4] if the utility be of the form $u(c) = \frac{c^{1-\gamma}}{\gamma}$ we can solve for $c^1(s_0)$