Macroeconomic Theory **V.V.Chari**
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Lecture 3*

One Sector Growth Model

Let s^t denote the partial history of events, where s_t takes value form a finite set $S = \{s^1, \ldots, s^S\}$. Allocations are given by $c(s^t), k(s^t), l(s^t)$. Let $z(s^t)$ denote the productivity shocks. Consider the following planning problem

$$\max_{\{c(s^t),\, l(s^t),\, k(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) U(c(s^t), l(s^t)) \tag{1}$$

$$\begin{aligned} \text{s.t.} \qquad c(s^t) + k(s^t) &\leq z(s^t) F(k(s^t), l(s^t)) + (1-\delta) k(s^{t-1}) \\ c(s^t) &\geq 0 \\ k(s^t) &\geq 0 \end{aligned}$$

The above setup is called *One-Sector Neoclassical Growth Model.* [1]

Question Why is this model attractive?
Answer

- perhaps the most important reason is that *National Income and Product Account* comes explicitly in this form

Product Side	Income Side
Private consumption	Compensation of employees
Private investment	Corporate profit
Government consumption	Rent
Government investment	Net interest
Net exports	Proprietor's income

Recall the National Account Identity

$$C + I + G + NX = Y$$

The right hand side of the feasibility constraint in the above model is exactly the right hand side of the national account identity (product side). Also note in competitive market $r_t = F_k$ and $w_t = F_l$ where r_t is the rental rate and w_t is the wage and F_l and F_k are marginal product of labor and capital. Let $F(.,.)$ be constant return to scale and we have

$$Y_t = F(k_t, l_t) = r_t k_t + w_t l_t$$

Which is the Income side in NIPA.

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[1] Note that in the presented model there is no restriction on the sign of investment, $k(s^t) \geq (1-\delta)k(s^{t-1})$. However, an alternative description of the world would be to assume, "explicitly", that once the capital is installed, it can not be consumed, i.e., investment is non-negative.

- Studies have concluded that fluctuations in z are large; similar in order of magnitude to those of $y = zF(k,l)$.
- We have data on k and l. Assuming a functional form for $F(.,.)$ the model allow us to calculate z.
- Understanding this simple model is essential to understand further elaboration.

Stationary Environment

In applied situations, we want "stationary" versions of model. Stationary environment in very loose terms means that future "looks like" past. And by that we mean that probabilistic laws that govern the uncertainty in the model does not change. This is very important for two reasons. First, as we'll making this assumption enable us to characterize formal solutions to many problems (which are not tractable in non-stationary environment). Second, without the assumption of stationarity estimation and testing the models is essentially impossible.[2] Hence, we now impose restrictions in this sense over $\mu(s^t)$: we will in general suppose it is a Markov process. Then, the probability measure can be rewritten as:

$$\mu(s^t) = \mu(s^t|s^{t-1})\mu(s^{t-1}),$$

where $\mu(s^t|s^{t-1})$ is time-invariant and $\mu(s_t|s_{t-1}) = \mu(s^t|s^{t-1})$.

Example 1: AR(2). Recall an AR(2) process is one that has the form:

$$\theta_t = c_1\theta_{t-1} + c_2\theta_{t-2} + \epsilon_t, \tag{2}$$

with $\theta_t \in \mathbb{R}$ and $\epsilon_t \sim i.i.d.$. Thus, (2) can be rewritten as:

$$\begin{bmatrix} \theta_t \\ \theta_{t-1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_{t-1} \\ \theta_{t-2} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix},$$

where we call $s_t = \begin{bmatrix} \theta_t \\ \theta_{t-1} \end{bmatrix}$.

We would also like allocations to be "stationary" functions of suitable state variables. Meaning that values of $c(.)$, $k(.)$ and $l(.)$ only depend on the realization of s^t.

Recursive Formulation

Let the current stock of capital and realization of current shock, (k, s), be the state variable and $c(k, s)$, $k'(k, s)$, $l(k, s)$ be the allocation functions. We want these functions to depend on the current period's information (k, s) and not on the information of periods before that one, since those do not contain relevant information. Our goal is to write the sequential problem (1) in recursive form.

[2] There is a huge literature in economterics on problems associated with non-stationarity. Technical discussion aside, the source of the trouble is that future doesn't "look like" past in non-stationary environment.

How to do this? An educated guess (at this point!) should be to write the problem as the following

$$v(k,s) = \max_{c,k',l}\left\{u(c,l) + \beta\sum_{s'}\mu(s'|s)v(k',s')\right\}$$

$$\begin{array}{lrcl} \text{s.t.} & c + k' & \leq & zF(k,l) + (1-\delta)k \\ & c & \geq & 0 \\ & k' & \geq & 0 \\ & 1 \geq l & \geq & 0 \end{array}$$

QuestionWhen is $v(k,s)$ is equal the utility in the sequential problem (1)? Are the policy functions resulting from the Dynamic Programming (DP) identical to those resulting from the Sequential form?

We are going to show that under relatively mild conditions the answer to the above questions are "always".

General Setup

Notations

Basically we have three objects of interest

- $x \in X$ is the (exogenous) *state* of the system. When $\mathcal{X}$ is σ-algebra *genrated* by X we call $(X, \mathcal{X})$ a *measurable space.* In general $(X, \mathcal{X})$ can be very abstract but for most applications we assume $X \subset \mathbb{R}^n$ with *Borel* σ-algebra . In the growth model $x = k$ is the current stock of capital.
- $y \in Y$ is set of choices or *controls* or *policies.* In the growth model $y = \{c, k', l\}$.
- $z \in Z$ is an exogenous stochastic event with measure space $(Z, \mathcal{Z}, \mu)$. where $\mathcal{Z}$ is the σ-algebra generated by Z and μ is the *probability measure* over Z. Sometimes we refer to it as *exogenous state variable.*

A Planner has the following preferences

$$\sum_t \beta^t \sum_{z^t} F(x_t, y_t, z_t)\mu(z^t)$$

Where $\mu(z^t)$ is probability distribution over history $z^t$and $1 \geq \beta \geq 0$ is the discount factor. $F : X \times Y \times Z \to \mathcal{R}$ is a *measurable*, real-valued function. It is sometimes called *one period return function.*

The feasible set of choices is $\Gamma : X \times Z \to X$ and it is assumed to be a *measurable correspondence.* It is also called *Policy Correspondence.* In the growth model policy correspondence is $\Gamma(k) = \{c, k', l \mid c+k' \leq zF(k,l)+(z-\delta),\ c \geq 0,\ k' \geq 0\ \}$.

From now on let $S = X \times Z$, $s = (x, z)$ and $\mathcal{S} = \mathcal{X} \otimes \mathcal{Z}$.

A *plan* is a value $\pi_0 \in X$ and a sequence of $\mathcal{Z}^t$-measurable functions

$$\pi_t : z^t \to x$$

A *feasible plan* from $s_0 \in S$ is a plan π_t with the following two properties

1. $\pi_0 \in \Gamma(s_0)$
2. $\pi_t(z^t) \in \Gamma(\pi_{t-1}(z^{t-1}, z_t)$

   measurable with respect to $\mathcal{Z}^t$, is feasible from $x_0 \in S$ (where $S : X \times Z$) if

$$\pi_t(s^t) \in \Gamma(\pi_{t-1}(z^{t-1}), z_t).$$

Question Does a feasible plan exist?
Answer The answer is yes, under very weak conditions as we are going to present.

Assumption 9.1 (SLP): *Suppose Γ is non-empty valued and has measurable selection, i.e. there exists a measurable function $h : S \to X$ s.t. $h(s) \in \Gamma(s) \ \ \forall s \in S$*

Under this assumption a feasible plan exists.[3]

Some intuition regarding this assumption: If a function varies "too" much, it might be non-measurable. This occurs because it might not be possible to "predict" the future in an accurate way using current information, since the variability of events is too large[4]. Information available at each time is presented by the σ-algebra. [5] Hereafter, we will continue to assume stochastic stream of exogenous events follow a Markov process. Hence, we define Q as a transition function on (Z,$\mathcal{Z}$):

Definition: *Let (Z,$\mathcal{Z}$) be a measurable space. A* ***transition function*** *is a function $Q : Z \times \mathcal{Z} \to [0, 1]$ such that:*

- *for each $z \in Z$, $Q(z, \cdot)$ is a probability measure on (Z,$\mathcal{Z}$)*
- *for each $A \in \mathcal{Z}$, $Q(\cdot, A)$ is a $\mathcal{Z}$-measurable function.*

[3] See Lemma 9.1 in SLP, page 243

[4] Note that this idea goes beyond the notion of (dis)continuity

[5] Consider the following example: suppose there are three state of the world, $Z = \{Sunshine, Rain, Snow\}$. Consider the following σ-algebra $=\{\{Sunshine\}, \{Rain, Snow\}, \emptyset, \{Sunshine, Rain, Snow\}\}$. This means that agent knows when it is Sunshine, but can not distinguish between Rain and Snow. Now suppose the endowment process, $y(.)$, is as follows

- $y(\{Sunshine\}) = 0$
- $y(\{Rain\}) = 1$
- $(\{Snow\}) = 2$

Now we see that the endowment process is not a measurable function in the sense that it varies "too much" over the event $\{Rain, Snow\}$. Agent can not recognize the two state therefore his optimal decision can not differ over the event $\{Rain, Snow\}$, yet he receives different endowment, so how should he make a selection out of the set $\{c|c \leq y(z)\}$? By the measurability assumption we prevent this types of confusions to happen.

We may interpret such a function as the *conditional probability* of event A happening next period given that currently a is realized:

$$Q(a, A) = Pr(z' \in A | z = a).$$

Assumption 9.2 (SLP): *$F : A \to \mathbb{R}$ is $\mathcal{A}$-measurable, ($\mathcal{A} = \mathcal{X} \otimes \mathcal{X} \otimes \mathcal{Z}$) and either:*

- *$F \geq 0$ or $F \leq 0$.*
- *For each $(x_0, z_0) = s_0 \in S$, and each plan $\pi \in \Pi(s_0)$,*

  $$F[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t]$$

  is $\mu^t(z_0, \cdot)$-integrable, t=1,2,...; and the limit

  $$F[x_0, y_0, z_0] + \lim_{n\to\infty} \sum_{t=1}^{n} \int_{Z^t} F[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t]\mu^t(z_0, dz^t)$$

  exists (although it may be plus or minus infinity).

In sequence form, this problem would be written as:

$$(SP) \;\; v^*(s) = \sup_{\pi \text{ feasible}} \{F[x_0, y_0, z_0] + \lim_{n\to\infty} \sum_{t=1}^{n} \int_{Z^t} F[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t]\mu^t(z_0, dz^t)\}$$

Let π^* denote the optimal plan under SP. And once again consider the recursive problem

$$(DP) \quad v(s) = v(x, z) = \sup_{y \in \Gamma(x,z)} \left[F(x, y, z) + \beta \int v(y, z')Q(z, dz') \right]$$

And let G be the *optimal policy correspondence* that solves DP:

$$G : \left\{ y \in \Gamma(x, z) : v(x, z) = F(x, y, z) + \beta \int v(y, z')Q(z, dz') \right\}$$

A plan is *generated by G from s_0* if

- $\pi_0 = g_0(s_0)$
- $\pi_t = g_t[\pi_{t-1}(z^{t-1}, z_t]$, all $z^t \in Z^t, t = 1, 2, \ldots$

where $g_0, g_1, \ldots$ is a sequence of measurable selection of G.

The following theorems establish the equivalence of two problems.

Theorem 9.2 (SLP): *Let $(X, \mathcal{X})$, $(Z, \mathcal{Z})$, Q, Γ, and β be given. Let Assumptions 9.1 and 9.2 hold, and v^* be defined by (SP). Let v be a measurable function satisfying the functional equation (DP), such that*

$$\lim_{t\to\infty} \int_{Z^t} \beta^t v[\pi_{t-1}(z^{t-1}), z_t]\mu^t(z_0, dz^t) = 0 \tag{3}$$

all $\pi \in \Pi$, all $(x_0, z_0) = s_0 \in S$.
Let G be the optimal policy correspondence that solves (DP), and suppose that G is non-empty and permits a measurable selection. Then $v = v^*$, and any plan π^* generated by G attains the supremum in (SP).

An important assumption for the next theorem is:

Assumption 9.3 (SLP): *If F takes on both signs, there is a collection of non-negative, measurable functions ($L_t : S \to \mathbb{R}_+$), t=0, 1, . . . , such that for all $\pi \in \Pi(s_0)$ and all $s_0 \in S$*

$$\begin{aligned} |F(x_0, y_0, z_0)| &\leq L_0(x_0); \\ |F[\pi_{t-1}(z^{t-1}), z_t]| &\leq L_t(s_0), \text{ all } z^t \in Z^t, t = 1, 2, ...; \end{aligned}$$

and

$$\sum_{t=0}^{\infty} \beta^t L_t(s_0) < \infty.$$

Theorem 9.4 (SLP): *Let $(X, \mathcal{X})$, $(Z, \mathcal{Z})$, Q, Γ, and β be given. Let Assumptions 9.1 - 9.3 hold, and v^* be defined by (SP). Let v be a measurable function satisfying the functional equation (DP) with optimal policy G . Assume that G is non-empty and permits a measurable selection. Let $(x_0, z_0) = s_0 \in S$, and let $\pi^* \in \Pi(s_0)$ be a plan that attains the supremum in (SP) for initial condition s_0. Then, there exists a plan π^G generated by G from s_0 such that*

$$\pi_0^G = \pi_0^*, \text{ and}$$

$$\pi_t^G(z^t) = \pi_t^*(z^t), \ \mu^t(z_0, \cdot) - a.e., \ t = 1, 2, ...$$

Now we assume $F(.,.,.)$ is bounded such that assumption 9.3 is satisfied and (DP) is well defined. Next we want to establish the existence of solution for (DP) and derive the properties of the solution. Consider the following mapping of functions:

$$Tf(x,z) = \sup_{y\in\Gamma(x,z)} \left\{ F(x,y,z) + \beta \int F(y,z')Q(z,dz') \right\}$$

Assumption 9.4 (SLP): *X is a convex Borel set in ($\mathbb{R}^l$), with its Borel subsets $\mathcal{X}$.*

Assumption 9.5 (SLP): *One of the following conditions holds:*

- *Z is a countable set and $\mathcal{Z}$ is the σ-algebra containing all subsets of Z: or*
- *Z is a compact (Borel) set in $\mathbb{R}^k$, with its Borel subsets $\mathcal{Z}$, and the transition function Q on $(Z, \mathcal{Z})$ has the Feller property.*[6]

Assumption 9.6 (SLP): *The correspondence $\Gamma : X \times Z \to X$ is nonempty, compact-valued, and continuous.*

Assumption 9.7 (SLP): *The function $F : A \to \mathbb{R}$ is bounded and continuous, and $\beta \in (0, 1)$.*

Theorem 9.6 (SLP): *Let $(X, \mathcal{X})$, $(Z, \mathcal{Z})$, Q, Γ, and β satisfy Assumptions 9.4 - 9.7, and the operator T on $C(S)$ by*

$$(Tf)(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int F(y, z')Q(z, dz') \right\}. \tag{4}$$

Then, $T : C(S) \to C(S)$; T has a unique fixed point v in $C(S)$ and for any $v_0 \in C(S)$,

$$||T^n v_0 - v|| \leq \beta^n ||v_o - v||, \quad n = 1, 2....$$

Moreover, the correspondence $G : S \to X$ defined by

$$G : \left\{ y \in \Gamma(x, z) : v(x, z) = F(x, y, z) + \beta \int v(y, z')Q(z, dz') \right\} \tag{5}$$

is nonempty, compact-valued and u.h.c.

Proof:
As an exercise show that under the assumptions above T is a maps space of bounded continuous functions on S into itself and satisfies *Blackwell Sufficient Conditions* and therefore is a *Contraction Mapping.* Then the existence and uniqueness of $v(s)$ follows from contraction mapping theorem. One can then apply *Berg's Theorem of Maximum* to establish that G is non-empty, compact-valued and u.h.c.

Feller Property

A transition function Q has the *Feller Property* if the following operator

$$(Tf)(z) = \int f(z')Q(z, dz'), \qquad \forall z \in Z$$

maps the space bounded continuous functions on Z into itself. Markov operators with this property are also called *stable.*

Example 2: Violation of Feller Property. Consider the following function:

[6] Q has the Feller Property if for all continuous function q, $h(z) = \int q(z)Q(z, dz')$ is continuous.

$$Q(z, z') = \begin{cases} \frac{1}{4} & w.p.1 \quad z < \frac{1}{2} \\ 1 & w.p.1 \quad z \geq \frac{1}{2} \end{cases}$$

consider the following continuous function

$$g(z) = \begin{cases} 0 & z < \frac{1}{4} \\ z - \frac{1}{4} & z \geq \frac{1}{4} \end{cases}$$

therefore when $z < \frac{1}{2}$

$$\begin{aligned} \int g(z')Q(z, dz') &= \int_0^{\frac{1}{4}} g(z)Q(z, dz') + \frac{1}{4}\int_{\frac{1}{4}}^1 (z - \frac{1}{4})dz' \\ &= \frac{1}{4} \times \frac{9}{32} \end{aligned}$$

and when $z \geq \frac{1}{2}$

$$\begin{aligned} \int g(z')Q(z, dz') &= \int_0^{\frac{1}{4}} g(z)Q(z, dz') + \int_{\frac{1}{4}}^1 (z - \frac{1}{4})dz' \\ &= \frac{9}{32} \end{aligned}$$

Thus $(Tg)(z)$ is not continuous at $z = 1/2$.