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Lecture 4*

Inheritance properties

Given the following dynamic programming problem

$$v(x,y) = \sup_{y\in\Gamma(x,z)} \left\{ F(x,y,z) - \beta \int v(y,z')Q(z,dz') \right\},$$

we will prove the following properties:

- F strictly increasing $\Longrightarrow v$ strictly increasing;
- F strictly concave $\Longrightarrow v$ strictly concave;
- F differentiable $\Longrightarrow$ v differentiable.

Before we prove the first result need the following assumptions in place

Assumption 9.8 (SLP) *For each $(y,z) \in X \times Z$, $F(.,y,z)$ is an strictly increasing function of x.*

Assumption 9.9 (SLP) *For each $z \in Z$, $\Gamma(.,z) : X \rightarrow X$ is increasing, i.e., $x \leq x' \Rightarrow \Gamma(x,z) \subseteq \Gamma(x',z)$*

As an example consider the following

$$c + h' \leq f(x,h) + (1-\delta)k,$$

so if k increases, the set of feasible allocations grows.

Lemma 9.5 *Let $(X,\mathcal{X})$, $(Z,\mathcal{Z})$ and Q, satisfy the assumptions 9.4 and 9.5. If $f : X \times Z \rightarrow \mathbb{R}$ is bounded and continuous, then Mf defined by*

$$(Mf)(y,z) = \int f(y,z')Q(z,dz'), \qquad \forall (y,z) \in X \in Z$$

is also. If f is (strictly) increasing in each of its first l arguments, then so is Mf. If f is (strictly) concave jointly in its first l arguments, then so is Mf.

Theorem 9.7 (SLP) *Let $(X,\mathcal{X}),(Z,\mathcal{Z}),Q,\Gamma,F$ and β satisfy the assumption 9.4-9.9, and let v be the unique fixed point of the operator T defined in theorem 9.6. Then for each $z \in Z$, $v(.,z) : X \rightarrow \mathbb{R}$ is strictly increasing.*

Proof

*These notes are prepared by Laurence Ales, Roozbeh Hosseini and Miguel Ricaurte. They are preliminary and possibly contain errors. Comments and feedbacks are welcome.

Note that set of bounded continuous functions on S that are non-decreasing is a closed subset of $C(S)$ and therefore is a complete metric space. Therefore, v must be a non-decreasing function (since it is a fixed point of T and T is an operator on $C(S)$)[1]. But under assumptions 9.8 and 9.9, lemma 9.5 implies that Tv is strictly increasing (a function that is result of adding an increasing function with a strictly increasing function, is strictly increasing). And since v is the fixed point, we must have $v = T(v)$. This finishes the proof.

Next we establish the strict concavity of v.

Assumption 9.10 (SLP) *For each $z \in Z$, $F(.,.,z)$ satisfies*

$$F[\theta(x,y) + (1-\theta)(x,y), z] \geq \theta F(x,y,z) + (1-\theta)F(x,y,z) \qquad \forall \theta \in (0,1)$$

Assumption 9.11 (SLP) *For all $z \in Z$ and all $x, x' \in X$, $y \in \Gamma(x,z)$ and $y' \in \Gamma(x',z)$ implies*

$$\theta y_(1-\theta)y' \in \Gamma[\theta y + (1-\theta)x', z] \qquad \forall \theta \in [0,1]$$

Theorem 9.8 (SLP) *Let $(X, \mathcal{X}), (Z, \mathcal{Z}), Q, \Gamma, F$ and β satisfy the assumption 9.4-9.7 and 9.10-9.11, and let v be the unique fixed point of the operator T defined in theorem 9.6; and G be the optimal policy correspondence. Then for each $z \in Z$, $v(.,z) : X \to \mathbb{R}$ is strictly concave and $G(.,z) : X \to X$ is a continuous (single-valued) function .*

Proof

Set of bounded continuous functions on S that are weakly concave in their first l arguments is a closed subset of $C(S)$ and therefore is a complete metric space. Thus, v must be a weakly concave function (since it is a fixed point of T and T is an operator on $C(S)$)[2]. But under assumptions 9.10 and 9.11, lemma 9.5 implies that Tv is strictly concave (a function that is result of adding a weakly concave function with a strictly concave function, is strictly concave). And since v is the fixed point, we must have $v = T(v)$. This finishes the proof of v being strictly concave. It follows then, immediately, by *Berg's Theorem of Maximum* that G is a single valued function and therefore is a continuous one.

Some intuition for this proof (and proof of previous theorem) can be given as the following:

We observe that T maps strictly concave functions into the set of strictly concave functions (Show it as an exercise). We define successive iteration of the operator

[1] See Corollary 1 of Theorem 3.2 (SLP), page 52

[2] See Corollary 1 of Theorem 3.2 (SLP), page 52

T as

$$\begin{aligned} T^1 v_0 &= T(v_0), \\ T^2 v_0 &= T(T(v_0)), \\ T^3 v_0 &= T(T^2(v_0)), \end{aligned}$$

the fixed point v^* will then be given by

$$v^* = \lim_{N\to\infty} T^N v_0, \qquad \forall v_0.$$

under this formulation it is clear that strict concavity may not be preserved at the limit[3]. The solution to this problem is to show that operator T maps concave functions into strict concave functions (show it as an exercise), so that, given the relation $Tv^* = v^*$, we must have that v^* is strictly concave.

Assumption 9.12 (SLP) *For each fixed $z \in Z$, $F(.,.,z)$ is continuously differentiable in (x, y) on the interior of graph of* Γ.

Theorem 9.10 (SLP) *Let $(X, \mathcal{X}), (Z, \mathcal{Z}), Q, \Gamma, F$ and β satisfy the assumption 9.4-9.7 and 9.10-9.12, and let v be the unique fixed point of the operator T defined in theorem 9.6; and $g = G$ be the optimal policy function. If $x_0 \in int(X)$ and $g(x_0, z_0) \in int(\Gamma(x_0, z_0))$, then $v(., z_0)$ is continuously differentiable in x at x_0, with derivatives given by*

$$v_i(x_0, z_0) = F_i[x_0, g(x_0, z_0), z_0], \qquad i = 1, \ldots, l$$

Proof

First note that the assumptions of theorem 9.8 are satisfied and v is an strictly increasing bounded continuous function. Since $x_0 \in int(X)$ and $g(x_0, z_0) \in int(\Gamma(x_0, z_0))$, there is some open neighborhood D of x_0 such that $g(x_0, z_0) \in int(\Gamma(x, z_0))$, for all $x \in D$. Define the function $W : D \to \mathcal{R}$

$$w(x) = F[x, g(x_0, z_0), z_0] + \beta \int v[g(x_0, z_0), z']Q(z_0, dz')$$

Then clearly $w(x) \geq v(x, z_0)$ for all $x \in D$ (why?), and equality is strict at x_0. Also $w(x)$ is strictly concave and differentiable in x (since $F(.,., z_0)$ is). Hence the *Benveniste and Sheinkman Theorem* [4]applies and we get the desired result.

The intuition for proof is as follows. Consider the growth model

$$\begin{aligned} v(k) &= \max_{c,k'} \left\{ u(c) + \beta v(k') \right\}, \\ s.t. \qquad c + k' &\leq f(k) + (1-\delta)k \end{aligned}$$

[3] $\{x^{1+\frac{1}{n}}\}_{n=1}^{\infty}$ is a sequence of strictly concave functions that converges to $f(x) = x$ (weakly concave)

[4] Theorem 4.10 (SLP), page 84

Suppose we want to establish differentiability in $\hat{k}$, let optimal policy at $\hat{k}$ be $k' = g(\hat{k})$. Now fix k' in a naiborhood of $\hat{k}$. In other words, consider an alternative policy $h(k) = k' = g(\hat{k})$. Clearly this policy is sub-optimal for $k \neq \hat{k}$. Now $w(k)$ is

$$w(k) = \left\{u\left(f(k) + (1 - \delta k - g(\hat{k})\right) + \beta v(\hat{k})\right\}$$

then the following are true

1. $w(\hat{k}) = v(\hat{k})$;
2. $w(k) \leq v(k), \quad \forall k$;
3. u is strictly concave implies w is strictly concave;
4. u is differentiable implies w is differentiable.

Then we use the result of Rockafellar[5] that state that the derivatives of both value functions are the same, so that they are both differentiable.

Now it remains to establish properties of value function v with respect to z. But we need the following assumptions.

Assumption 9.13 (SLP) *For each $(x, y) \in X \times X$, $F(x, y, .)$ is an strictly increasing function of z.*

Assumption 9.14 (SLP) *For each $x \in X$, $\Gamma(X, .) : Z \to Z$ is increasing, i.e., $z \leq z' \Rightarrow \Gamma(x, z) \subseteq \Gamma(x, z')$.*

Assumption 9.15 (SLP) *Q is monotone; that is, if $f :\to \mathbb{R}$ is non-decreasing, then the function $(Mf)(z) = \int f(z')Q(z, dz')$ is also non-decreasing* [6].

Theorem 9.11 (SLP) *Let $(X, \mathcal{X}), (Z, \mathcal{Z}), Q, \Gamma, F$ and β satisfy the assumption 9.4-9.7 and 9.13-9.15, and let v be the unique fixed point of the operator T defined in theorem 9.6. Then for each $x \in X$, $v(x, .) : Z \to \mathbb{R}$ is strictly increasing .*

Proof

The logic of proof is exactly the same as proof the of theorems 9.10 and 9.8. We only need to establish that if $f(x, .) : Z \to \mathbb{R}$ is non-decreasing function of z (fixed x), $(Tf)(x, .)$ is strictly increasing. Let $z_1 < z_2$ and $y_1 \in \Gamma(x, z_1)$. Then

$$\begin{aligned}
(Tf)(x, z_1) &= F(x, y_1, z_1) + \int f(y_1, z')Q(z_1, dz') \\
&< F(x, y_1, z_2) + \int f(y_1, z')Q(z_2, dz') \qquad \text{(by assumption 9.15 and 9.13)} \\
&\geq max_{y \in \Gamma(x, z_2)} \left\{F(x, y, z_2) + \int f(y, z')Q(z_2, dz')\right\} \\
&= (Tf)(x, z_2) \qquad \text{(by assumption 9.14)}
\end{aligned}$$

[5] Rockafellar 1970 , *Convex Analysis*, theorem 25.1, p.242

[6] This property is also called *first order stochastic dominance* in the particular case with $z \in \mathbb{R}$

Exercise complete the argument of proof.

Unbounded Returns

We now wish to consider a constant return to scale environment where the following assumptions hold

Assumption 9.18 (SLP) *$X \subset \mathbb{R}^l$ is a convex cone, with its borel subsets $\mathcal{X}$.*
Assumption 9.19 (SLP) *The correspondence $\Gamma : X \times Z \rightarrow X$ is non-empty, compact-valued, and continuous; and for any $(x, z) \in X \times Z$,*

$$y \in \Gamma(x, z) \Rightarrow \lambda y \in \Gamma(\lambda x, z) \qquad \forall \lambda$$

and for some $\alpha \in (0, \beta^{-1})$

$$||y||_l \leq \alpha ||x||_l \qquad \forall y \in \Gamma(x, z) \quad \forall (x, z) \in X \times Z$$

Assumption 9.19 (SLP) *The function F is continuous over graph of Γ; for each $z \in Z$, the function $F(.,.,z)$ is homogeneous of degree one; for some $0 < B < \infty$,*

$$|F(x, y, z)| \leq B(||x||_l + ||y||_l) \qquad \text{for all (x,z,y) in graph of } \Gamma$$

and $0 < \beta < 1$.

The following exercise shows the principle of optimality holds under these assumptions.

Exercise Show that under assumptions 9.5 and 9.18-9.20, assumption 9.1-9.3 hold[7].

Decentralizing the growth model

We start from the following sequential problem

$$\max_{c(s^t)} \sum_t \beta^t \sum_{s^t} \mu(s^t) u\left(c(s^t)\right) \tag{1}$$
$$s.t. c(s^t) + k(s^t) \leq z(s^t) F(k(s^{t-1}), l(s^{t-1})) + (1 - \delta) k(s^{t-1}), \quad \forall t, \forall s^t,$$
$$c(s^t) \geq 0, \quad k(s^t) \geq 0,$$
$$0 \leq l(s^t) \leq 1.$$

If we look at the Arrow Debreu definition, the equilibrium consists of be defined allocations $c(s^t), l(s^t), k(s^t)$ and a price system; $q(s^t)$ (A-D price of consumption), $q(s^t)w(s^t)$ (wage rate) and $q(s^t)r(s^t)$ (rental rate). In an Arrow-Debreu equilibrium, households solve the following problem taking price system (described above) as given

[7] This is exercise 9.8 in SLP, page 271

$$\max_{c(s^t)} \sum_t \beta^t \sum_{s^t} \mu(s^t) u\left(c(s^t)\right) \tag{2}$$

$$s.t. \sum_{t,s^t} q(s^t)\left(c(s^t)+k(s^t)\right) \leq \sum_{t,s^t} q(s^t)\left(z(s^t)k(s^{t-1}) + w(s^t)l(s^t) + (1-\delta)\, k(s^{t-1})\right),$$

$$c(s^t) \geq 0, \quad k(s^t) \geq 0,$$

$$0 \leq l(s^t) \leq 1.$$

Firms in this economy will be solving the following problem (again taking prices as given)

$$\max_{k(s^t),l(s^t)} z(s^t)F(k,h) - w(s^t)l(s^t) - r(s^t)k(s^t).$$

The Arrow-Debreu Competitive Equilibrium as before is defined as an allocation $c(s^t), l(s^t), k(s^t)$ and a price system $q(s^t), q(s^t)w(s^t), q(s^t)r(s^t)$ so that both the Household and the firm solve the above problems taking prices as given and such that market clears, we will have

$$\left(l(s^t), k(s^{t-1})\right) \in \arg\max z(s^t)F(k,l) - w(s^t)l(s^t) - r(s^t)k(s^t), \quad \forall t, \forall s^t.$$

Note that firms decide allocations after they have observed the productivity shock.

Question Is it possible to set up the problem as a sequence of sequential market, that has a recursive formulation?

Sequential budget constraint of the household is the following

$$c(s^t) + k(s^t) \leq r(s^t)k(s^{t-1}) + (1-\delta)\, k(s^{t-1}) + w(s^t)l(s^t), \quad \forall t, \forall s^t,$$

$$c(s^t) \geq 0, \quad k(s^t) \geq 0,$$

We now assume that the random variable s follows a Markov Process. This allows us to right the sequential market problem in a recursive form.

Recursive competitive Equilibrium (RCE)

consider the following Dynamic Programming problem faced by each household

$$\begin{aligned} v(k,s,K) &= \max_{c,k',l} \left\{ u(c) + \beta \int v(k',s',K')Q(s,ds') \right\}, \\ c + k' &\leq r(K,s)k + (1-\delta)\, k + w(K,s)l, \\ K' &= G(K,s). \end{aligned}$$

where K is aggregate stock of capital.

A *Recursive Competitive Equilibrium*[8] is

i) A value function v;

[8] See also Mehra and Prescott,"Recursive competitive Equilibrium: the Case of homogeneous Households",*Econometrica*, Vol. 48, No. 6. (Sep., 1980), pp. 1365-1379.

ii) A set of policy functions $c(k,s,K)$, $k' = g(k,s,K)$, $l(k,s,K)$;

iii) A set of pricing functions $r(K,s)$, $w(K,s)$;

iv) an aggregate law of motion $G(k,s)$;

such that,

a) v solves the dynamic programming problem;

b) the policy functions $c(k,s,K)$, $k' = g(k,s,K)$, $l(k,s,K)$ attain the v;

c) $r(K,s)$, $w(K,s)$ satisfy the firm's optimality

$$\begin{aligned} z(s)F_k(k,1) &= r(k,s), \\ z(s)F_l(k,1) &= w(k,s). \end{aligned}$$

d) $g(K,s,K) = G(K,s)$.

e) Market clearing