Lecture 5*

Simple Version of the Growth Model

Consider the following one-sector growth model

$$\max_{\{c_t(s^t),\ k_{t+1}(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) u(c_t(s^t)) \tag{1}$$

$$\begin{aligned} \text{s.t.} \qquad c_t(s^t) + k_{t+1}(s^t) &\leq z(s^t) f(k_t(s^{t-1})) \\ c_t(s^t) &\geq 0 \\ k_{t+1}(s^t) &\geq 0 \end{aligned}$$

And assume that s_t follows a markov process with transition probability $Q(s, s')$. Assume the following

- $z(s)$ is strictly increasing in s, bounded away from zero and bounded above. Without loss of generality assume $z(s) \in [1, \bar{z}]$ for some $\bar{z} < \infty$
- $Q(.,.)$ is *monotone* and has *feller property*
- $u(.)$ is continously differentiable, strictly concave and strictly increasing and $0 < \beta < 1$
- $f(.)$ is continuously differentiable, weakly concave, and strictly increasing. Also, $f(0) > 0$[1], $f'(0) = \infty$ and $f'(\infty) = 0$.

The above prblem can be written in the following recursive form

$$V(k, s) = \max_{\substack{c + k' \leq z(s) f(k) \\ c \geq 0 \\ k' \geq 0}} \{u(c) + \beta \int V(k', s') Q(s, ds')\}$$

with solution $V(k, s)$ and policy functions $c = c(k, s)$ and $k' = g(k, s)$.

Claim I: $V(k, s)$ is strictly increasing in k and s
proof:

$$V(k, s) = \max_{k' \geq 0} \{u(z(s) f(k) - k') + \beta \int V(k', s') Q(s, ds')\}$$

Check that assumptions of theorem 9.7 and theorem 9.11 are satisfied.

*These notes are prepared by Laurence Ales, Roozbeh Hosseini and Miguel Ricaurte. They are preliminary and possibly contain errors. Comments and feedbacks are welcome.
[1] This is just a technical assumption and can be dispensed with

Claim II: $V(k,s)$ is strictly concave in k
proof:

$$V(k,s) = \max_{k' \geq 0}\{u(z(s)f(k) - k') + \beta \int V(k', s')Q(s, ds')\}$$

Check that assumptions of theorem 9.8 are satisfied.

Claim III: $g(k,s)$ and $c(k,s)$ are strictly increasing in k and s
proof: First order condition

$$u'(z(s)f(k) - g(k,s)) = \beta \int V_k(g(k,s), s')Q(s, ds')$$

Let $k_2 > k_1$ and suppose $g(k_2, s) \leq g(k_1, s)$. Since $V(., s)$ is strictly concave we have

$$\beta \int V_k(g(k_1,s), s')Q(s, ds') \leq \beta \int V_k(g(k_2,s), s')Q(s, ds')$$

therefore, first order condition implies

$$u'(z(s)f(k_1) - g(k_1,s)) \leq u'(z(s)f(k_2) - g(k_2,s))$$

Thus and strict concavity of $u(.)$ implies

$$\begin{aligned} z(s)f(k_1) - g(k_1,s) &\geq z(s)f(k_2) - g(k_2,s) \\ &\geq z(s)f(k_2) - g(k_1,s) \qquad \text{Since } g(k_2,s) \leq g(k_1,s) \end{aligned}$$

and Hence

$$z(s)f(k_1) \geq z(s)f(k_2)$$

Which implies

$$k_1 \geq k_2$$

a contradiction. Therefore, $g(k_2, s) > g(k_1, s)$. Strict concavity of $V(., s)$ thus implies

$$\beta \int V_k(g(k_1,s), s')Q(s, ds') > \beta \int V_k(g(k_2,s), s')Q(s, ds')$$

Thus, first order condition implies

$$u'(c(k_1,s)) > u'(c_2(k_2,s))$$

And by strict concavity of $u(.)$

$$c(k_1,s) < c(k_2,s)$$

Now let $s_2 > s_1$ and suppose $g(k, s_2) \leq g(k, s_1)$. Then strict concavity of $V(., s)$ implies

$$\beta \int V_k(g(k,s_1), s')Q(s, ds') \leq \beta \int V_k(g(k,s_2), s')Q(s, ds')$$

And by first order conditions

$$u'(z(s_1)f(k) - g(k,s_1)) \leq u'(z(s_2)f(k) - g(k,s_2))$$

By strict concavity of $u(.)$

$$\begin{aligned} z(s_1)f(k) - g(k,s_1) &\geq z(s_2)f(k) - g(k,s_2) \\ &\geq z(s_2)f(k) - g(k,s_1) \qquad \text{Since } g(k,s_2) \leq g(k,s_1) \end{aligned}$$

Hence

$$z(s_1)f(k) \geq z(s_2)f(k)$$

Which implies (since z(.) is strictly increasing)

$$s_1 \geq s_2$$

A contradiction. Proving that $c(k,s)$ is strcitly increasing in s can be done in the same fashion as above.

Claim IV: For every s, there exists a unique k^* such that

$$\begin{aligned} g(k,s) > k, \quad k < k^* \\ g(k,s) < k, \quad k > k^* \end{aligned}$$

Moreover, if $s_2 > s_1$ then, $k^*_{s_2} > k^*_{s_1}$.

proof: (Difficult!!)