

Lecture 6*

1 Search Models

We now introduce a search model (McCall-Lipmann). Consider a worker who is laid off periodically with probability ϑ ; while unemployed the worker searches for a job. Jobs are characterized by a wage w . While searching, new wage offers are distributed with density $f(w)$ on a support $[0, \bar{w}]$; note that a wage of zero denotes the worker being unemployed. In the model we will study we allow for a worker to quit, that is at any period the worker will have the choice of leaving the job and search for a better wage. As we shall see the search process is characterized by a reservation wage w^* : a level below which the worker refuses the job and draws another offer in the following period.

We now construct the sequential formulation for this problem. To model his decision process we introduce the indicator function d_t that at every t takes value 1 if he accepts the job, and 0 if he refuses. Uncertainty is given by the random variable z_t that takes value 1 if the worker is laid off and 0 if he retains his job. The worker problem will be given by

$$\begin{aligned} \max \sum_{t=0}^{\infty} \beta^t \mu(w^t, z^t) u(d_t(w^t, z^t) w_t), \\ \text{s.t. } d_t(w^t, z^t) = 0 \quad \text{if } z_t = 1. \end{aligned}$$

We now rewrite the above problem in recursive form. Considering a current wage offer w , if the worker accepts the offer his utility will be

$$u(w) + \beta [(1 - \vartheta) v(w) + \vartheta v(0)],$$

if he continues to search

$$0 + \beta \int_0^{\bar{w}} v(w') f(w') dw'.$$

The maximization problem will then be

$$v(w) = \max \left\{ u(w) + \beta [(1 - \vartheta) v(w) + \vartheta v(0)], \beta \int_0^{\bar{w}} v(w') f(w') dw' \right\}. \quad (1)$$

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Note how the operator associated with the Bellman equation introduced above maps continuous increasing functions in continuous increasing functions, then taking into account a value of $0 < \beta < 1$ we have that Blackwell conditions are satisfied. If we assume $u(0) = 0$ it is easy to show that

$$v(0) = \beta \int_0^{\bar{w}} v(w')f(w')dw' = A,$$

we can then rewrite equation (??) as

$$v(w) = \max \{u(w) + \beta [(1 - \vartheta)v(w) + \vartheta A], A\}. \quad (2)$$

Solving this recursive problem numerically we would obtain a shape for the value function as the one depicted in Figure 10.4 in SLP (pg. 307) we notice how in this case the value function is not concave; this is due to the fact that we are maximizing over a non convex set (stay, search); from the picture we also see how, if the worker is below a treshold given by w^* , he will search if the wage offer will be above he will stay; so

$$v(w) = \begin{cases} A & \text{if } w \leq w^*, \\ u(w) + \beta [(1 - \vartheta)v(w) + \vartheta A] & \text{if } w \geq w^*. \end{cases} \quad (3)$$

For continuity the following must hold

$$u(w^*) + \beta [(1 - \vartheta)v(w^*) + \vartheta A] = A.$$

From (??) if $w \geq w^*$ we have

$$v(w) = \frac{u(w) + \beta \vartheta A}{1 - \beta(1 - \vartheta)},$$

we can now substitute the above in the definition of A

$$\begin{aligned} A &= \beta \int_0^{w^*} v(w')f(w')dw' + \beta \int_{w^*}^{\bar{w}} \frac{u(w') + \beta \vartheta A}{1 - \beta(1 - \vartheta)} f(w')dw' = \\ &= \beta AF(w^*) + \dots \end{aligned}$$

so that

$$(1 - \beta)A = \frac{\beta}{1 + \beta\vartheta - \beta F(w^*)} \int_{w^*}^{\bar{w}} u(w')f(w')dw',$$

substituting A we get the a reservation wage that will solve the following

$$[1 + \beta\vartheta - \beta F(w^*)]u(w^*) = \beta \int_{w^*}^{\bar{w}} u(w')f(w')dw'.$$

The type of question we now would like to answer is what is the typical length of unemployment or what is the fraction of the unemployed population. In order to answer such questions it will be useful to introduce Markov processes. Recall that a stochastic process is a sequence of random variables σ_t defined on an increasing sequences of σ -algebras $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$; that is σ_t is \mathcal{F}_t -measurable for all t . In addition a Markov process has the additional property that

$$\Pr(\sigma_{t+1}|h_t) = \Pr(\sigma_{t+1}|s_t),$$

where we denoted with h_t the vector with partial histories up to time t : $h_t = (s_0, \dots, s_t)$. Going back to the original search model we have that the optimal policy function associated with problem (??) induces a Markov process for the wage.

For example we can consider the wage, after having observed ϑ , to be in one of the following set:

$$\begin{aligned} D &= \{0\}, \\ A^c &= (o, w^*], \\ A &= (w^*, \bar{w}]. \end{aligned}$$

starting from one of these set at time t , at time $t + 1$ we will be in one of the three with probabilities given by

$$\begin{aligned} \Pr(w' \in A|w \in A) &= (1 - \vartheta), \\ \Pr(w' \in D|w \in A) &= \vartheta, \\ \Pr(w' \in A^c|w \in D) &= F(w^*), \\ \Pr(w' \in A|w \in D) &= 1 - F(w^*), \\ \Pr(w' \in A^c|w \in A^c) &= F(w^*), \\ \Pr(w' \in A|w \in A^c) &= 1 - F(w^*). \end{aligned}$$

With the above probabilities we have completely described the transition function associated with the Markov process on wages induced by an optimal decision rule. The transition matrix will be given by

$$\Pi = \begin{array}{c} A \\ A^c \\ D \end{array} \begin{array}{ccc} A & A^c & D \\ \left(\begin{array}{ccc} 1 - \vartheta & 0 & \vartheta \\ 1 - F(w^*) & F(w^*) & 0 \\ 1 - F(w^*) & F(w^*) & 0 \end{array} \right) \end{array}$$

Note how having two rows and two columns equal would have enabled us to collapse A^c and D into one singular state. The above transition matrix will describe a Markov process on the probability distribution over the three states; so let $p_t = (p_t(A), p_t(A^c), p_t(D))$ we will then have

$$p_{t+1} = p_t \Pi, \tag{4}$$

that is

$$(p_{t+1}(A), p_{t+1}(A^c), p_{t+1}(D)) = (p_t(A), p_t(A^c), p_t(D)) \begin{pmatrix} \Pi \end{pmatrix},$$

Having defined the transition matrix Π we now want to derive the stationary probability distribution p^* that solves $p^* = p^*\Pi$. Suppose we begin with a probability distribution at time zero given by $p_0 = (\alpha_1, \alpha_2, \alpha_3)$ at time 1, following equation (??), we will have

$$\begin{aligned} p_1(A) &= \alpha_1(1 - \vartheta) + (\alpha_2 + \alpha_3)(1 - F(w^*)), \\ p_1(A^c) &= (\alpha_2 + \alpha_3)F(w^*), \\ p_1(D) &= \alpha_1\vartheta. \end{aligned}$$

Following chapter 11 in SLP we now look for some convergence results for p under a transition matrix Π .

Theorem 11.1 *Let $S = \{s_1, \dots, s_l\}$ be a finite set, and let the stochastic matrix Π define transition probabilities on S . Then*

- a) S can be partitioned into $M \geq 1$ ergodic sets and a transient set.
- b) The sequence $\left\{ \frac{1}{n} \sum_{k=0}^{n-1} \Pi^k \right\}_{n=1}^{\infty}$ converges to a stochastic matrix Q . That is, $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} p^k = p_0 Q$, for any sequence $\{p_k\} = \{p_0 \Pi^k\}$ where $p_0 \in \Delta^l$.
- c) Each row of Q is an invariant distribution, so $p_0 Q$ is an invariant distribution for each $p_0 \in \Delta^l$; and every invariant distribution for Π is a convex combination of the rows of Q .

The above theorem gives us a weak notion of convergence since it is only the average that converges.

Theorem 11.4 *Let $S = \{s_1, \dots, s_l\}$ be a finite set, and let the stochastic matrix Π define transition probabilities on S . For $n = 1, 2, \dots$, let $\varepsilon_j^{(n)} = \min_i \pi_{ij}^{(n)}$, $j = 1, \dots, l$ and let $\varepsilon^{(n)} = \sum_{j=1}^l \varepsilon_j^{(n)}$. Then S has a unique ergodic set with no cyclically moving subsets if and only if for some $N \geq 1, \varepsilon^{(N)} > 0$. In this case $\{p_0 \Pi^k\}$ converges to a unique limit $p^* \in \Delta^l$, for all $p_0 \in \Delta^l$, and convergence is at a geometric rate that is independent of p_0 .*

A sufficient condition for this result would require that $\Pi_{ij} > 0 \forall_{ij}$ or that exists a n such that $\Pi_{ij}^n > 0 \forall_{ij}$; that is we can transit from any state to any other in a finite number of steps

We now begin to study continuous state spaces, we begin with an example on capital accumulation; given the following FE problem

$$\begin{aligned} v(k, s) &= \max_{c, k'} \left\{ u(c) + \beta \int v(k', s') Q(s', ds') \right\} \\ c + k' &\leq z(s)f(k), \quad c \geq 0, k \geq 0 \end{aligned} \tag{5}$$

in the above formulation we could interpret the shock as $z(s)f(k) + (1 - \delta)k$, setting $\delta = 1$, or $z(s)(f(k) + 1 - \delta)$. Let $x = z(s)f(k)$ be the current output, and y the aggregate capital stock, we rewrite the maximization problem (??) as

$$v(k, s) = \max_{y \leq x'} \left\{ u(x - y) + \beta \int v(z(s')f(y))Q(ds') \right\}$$

since $Q(s', ds') = Q(ds')$ assuming that the shocks are i.i.d. Note how the expected output for tomorrow $\int z(s')f(g(x))Q(ds')$.

A transition function will be defined on the measurable space (S, \mathcal{L}) where $S = [0, \bar{x}]$. Our claim is that given $x' = z(s')f(g(x))$ the optimal policy function g and the underlying uncertainty induces a transition function on S . Suppose $F(s') = \Pr(s \leq s')$ then the uncertainty will be described by the conditional probability distribution $P(x, [0, \hat{x}]) = P(x' \in [0, \hat{x}] | x)$. Let $x' = \hat{x}$ if $\hat{x} = z(\hat{s})f(g(x))$ we define \hat{s} as $\hat{s} = z^{-1}\left(\frac{\hat{x}}{f(g(x))}\right)$ so that

$$P(x, [0, \hat{x}]) = F\left(z^{-1}\left(\frac{\hat{x}}{f(g(x))}\right)\right) = \int_0^\infty \hat{x} dF\left(z^{-1}\left(\frac{\hat{x}}{f(g(x))}\right)\right).$$

The transition function p introduces a Markov operator

$$(Th)(z) = \int h(z')p(z, dz'), \quad (6)$$

to this operator is associated it's adjunct operator defined by

$$(T^*\lambda)(A) = \int Q(z, A)\lambda(dz). \quad (7)$$

There is an invariant distribution if operator T^* has a fixed point. We say that T^* has a fixed point if for all $A \in \mathcal{L}$ we have

$$\lambda^*(A) = \int P(z, A)\lambda^*(dz).$$

We now ask under what rucola condition we have convergence to an invariant distribution.

Theorem 12.10 *If $S \subset R^l$ is compact and P has the Feller property, then there exists a probability measure that is invariant under P .*

We now introduce two additional property for our markov operators. We will say that T is **monotone** if $f \geq g$ then $\int f(s)P(s, ds') > \int g(s')P(s, ds')$; moreover we say that T has the **mixing** property (Assumption 12.1, SLP) if there exists $c \in S$, $\varepsilon > 0$ and $N \geq 1$ such that $P^N(a, [c, b]) \geq \varepsilon$ and $P^N(b, [a, c]) \geq \varepsilon$.

Theorem 12.12 *Let $S = [a, b] \subset R^l$. If P is monotone, has the Feller property and has the mixing property (Assumption 12.1, SLP), then P has a unique invariant probability measure λ^* , and $T^{*n}\lambda_0 \implies \lambda^*$, for all λ_0 .*

We now look at an example where the mixing condition does not hold.

$$\Pi = \begin{bmatrix} 0 & \Pi_1 \\ \Pi_2 & 0 \end{bmatrix},$$

we then have

$$\begin{aligned}\Pi^n &= \begin{bmatrix} (\Pi_1\Pi_2)^n & 0 \\ 0 & (\Pi_2\Pi_1)^n \end{bmatrix}, \\ \Pi^{n+1} &= \begin{bmatrix} 0 & (\Pi_1\Pi_2)^n \Pi_1 \\ (\Pi_2\Pi_1)^n \Pi_2 & 0 \end{bmatrix}.\end{aligned}$$

We see how Π^n does not converge, it actually only has convergent subsequences. Hence it has no invariant probability measures.

Definition *A subset $B \subseteq S$ is ergodic if for all $s \in B$, $P(s, B) = 1$, and for any proper subset $C \subseteq B$, $P(s, C) < 1$.*

Notice how the mixing condition guarantees that the whole space state is an ergodic set.