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Lecture 7

Incomplete Markets: Pure Currency Economy[1]

Let the economy have a limited access to assets. In particular, agents can only hold money balances as assets. Money does not yield any interest rate and for the time being we assume $p_t = \overline{p}\ \ \forall t$.
The state-dependent allocations in this economy are given by $c_t(z^t)$ and $m_t(z^t)$, the real balances (i.e. $m_t \equiv \frac{M_t}{p_t}$). The variable z_t represents a taste shock that increases demand for consumption good, where $z \in Z = [\underline{z}, \overline{z}]$. The budget constraint every agent faces is:

$$c_t(z^t) + m_t(z^t) \leq y + m_{t-1}(z^{t-1}) \tag{1}$$

The LHS contains the decision variables: current period's consumption and asset (money balances) holdings. The RHS contains the resources: y is the (exogenously determined) endowment and the money balances coming from the previous period.

Additionally, we impose a *cash-in-advance* constraint, accordingly with the incomplete markets assumption:

$$c_t(z^t) \leq m_{t-1}(z^{t-1}) \tag{2}$$

with $c_t(z^t) \geq 0$ and $m_{t-1}(z^{t-1}) \geq 0$.

Additionally, let the stochastic shocks, z_t, occur at an individual level such that there is no aggregate fluctuations. Let the economy have a large number of households who face the restrictions described above. Taken all the above into account, the consumer's problem is:

$$\max_{\{c_t(z^t),\ m_t(z^t)\}} \sum_{t=0}^{\infty} E\left[\beta^t U(c_t(z^t), z_t)\right] \tag{3}$$

$$\begin{aligned} \text{s.t.} \qquad c_t(z^t) + m_t(z^t) &\leq y + m_{t-1}(z^{t-1})\ \forall t \\ c_t(z^t) &\leq m_{t-1}(z^{t-1})\ \forall t \\ c_t(z^t) &\geq 0 \\ m_t(z^t) &\geq 0 \end{aligned}$$

This problem can be written in a recursive form as follows:

$$v(m, z) = \max_{c, m'} \left[U(c, z) + \beta \int v(m', z')\mu(dz')\right] \tag{4}$$

[1]This Lecture Note is a version of the model that appears in section 13.5, Ch. 13 of *Recursive Methods in Economic Dynamics*, Stockey and Lucas, 1989.

$$
\begin{aligned}
\text{s.t.} \qquad c + m' &\leq y + m \\
c &\leq m \\
c &\geq 0 \\
m &\geq 0
\end{aligned}
$$

The justification for each constraint has an equivalent explanation to the case for the Functional Equation. Additionally, note that μ is the probability measure on $(Z, \mathcal{Z})$, with $\mathcal{Z}$ the Borel sets of Z.

Before we can characterize the solution to this problem, note that as always, $\beta \in (0, 1)$ and $U : \mathbb{R} \times Z \to \mathbb{R}$ is bounded and continuously differentiable. For a given $z \in Z$, we assume $U(\cdot, z)$ is strictly increasing and strictly concave and that for a a $c \in \mathbb{R}_+$, $U_c(c, \cdot)$ is strictly increasing in z.

These assumptions result in policy functions $m' = g(m, z)$ and $c = c(m, z)$ that are continuous.

The *stationary equilibrium* of this economy is given by

1. A value function $v(m, z)$
2. Policy functions $g(m, z)$ and $c(m, z)$
3. A distribution function λ

such that:

1. $v(m, z)$, $g(m, z)$, and $c(m, z)$ solve the Dynamic Programming problem
2. A distribution function λ is stationary in equilibrium. This means that if tomorrow's distribution is:

$$\lambda(A) = \int \int \delta(m, z, a)\mu(dz)\delta(dm) \ \ ,\forall \text{ sets } A,$$

$$\delta(m, z, a) = \begin{cases} 1 & \text{if } g(m, z) \in A \\ 0 & \text{if not.} \end{cases}$$

3. Markets clear:

$$\int \int g(m, z)\mu(dz)\lambda(dm) = \int m\lambda(dm)$$
$$\int \int c(m, z)\mu(dz)\lambda(dm) = y.$$

The *invariant distribution* result is specially important in this model. Intuitively, it implies that the distribution of real balances among the population does not change through time. That is, the aggregate real balance holdings is constant in time - look at first "Markets Clear" condition - although individuals face an uncertain environment. Hence, it guarantees that a stationary equilibrium exists in which the price level is constant in time, as supposed originally.

The policy function $g(\cdot, \cdot)$ induces a Markov Process:

$$P(m,A) = \mu[z|gm, z \in A], \quad A \subseteq Z.$$

$$P(m,A) = \int 1_{\{g^{-1}(m,A)\}}\mu(dz') \tag{5}$$

where

$$1_{\{g^{-1}(m,A)\}} = \begin{cases} 1 & \text{if } z \in g_m^{-1}(A) \\ 0 & \text{if } z \notin g_m^{-1}(A). \end{cases}$$

Given a transition function $P(m,A)$, we may establish the conditions for the existence of an invariant distribution:

1. Markov Operator to preserve continuity. For all continuous bounded function,

$$(Tf)(m) = \int f(m')P(m, dm') \tag{6}$$

is continuous and bounded (Feller Property). Since $m' = g(m,z)$, (6) becomes:

$$(Tf)(m) = \int f(g(m,z))\mu(dz), \tag{7}$$

where we have used (5) and the fact that z is iid (today's value does not depend on previous realization of z). If g is a continuous function, then Tf is continuous as well. We require all these characteristics because we are dealing with a continuous state space.

2. Markov Operator is Monotone. If f is increasing, Tf is increasing as well (note that g being increasing follows from f being so).

3. Mixing Condition. Let the states rectangle be $[a,b]$. Then, there exists an $\epsilon > 0$ such that for any $c \in [a,b]$, for $N \geq 1$:

$$p^N(a,[c,b]) \geq \epsilon$$
$$p^N(b,[a,c]) \geq \epsilon,$$

where $p^N(\cdot)$ is the probability of lying in the interval after N iterations. This aims at guaranteeing that lower and upper bounds are not going to be the only solutions to the problem.

(missing graph)

Let:

$$g(\phi(\underline{z}), \underline{z}) = \phi(\underline{z})$$
$$g(\phi(z), z) = \phi(z)$$

$$g(c, \tilde{z}) = c,$$

where the last equality defines $\tilde{z}$. Let $[\tilde{z}, \underline{z}]$ be the set of shocks such that:

$$\begin{aligned} g(m, z) > m \quad \text{if} \quad m < c \\ g(m, z) > c \quad \text{if} \quad m > c. \end{aligned}$$

Assume there exists $\alpha > 0$ such that

$$\mu[z_1, z_2] \geq \alpha(z_2 - z_1) \ \ \forall z_1, z_2$$

the density if uniformly bounded away from zero. It is easy to show that there exists N_1 such that

$$p^{N_1}(y, [c, \phi(\underline{z})]) \geq \alpha^N(\tilde{z} - \underline{z})$$

A similar argument holds for $[\underline{z}, \tilde{z}]$. Thus, there exists an invariant distribution such that aggregate money holdings are constant in time.