Macroeconomic Theory **V.V.Chari**
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Lecture 8*

1 Huggett's Model

Let the consumer preferences be given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

We consider income as a random variable $w(s_t)$ over the finite state space $S = \{s^1, \ldots, s^N\}$ taking values respectively $\{w^1, \ldots, w^N\}$. We consider the stochastic process s_t to be a Markov process with transition probability over S given by

$$\mu_{ij} = prob\left(s_{t+1} = s^j | s_t = s^i\right)$$

this transition probability also gives us the fraction of the population subject to $s_{t+1} = s^j$ given $s_t = s^i$. This second interpretation of μ_{ij} enables us to see how in this model there is no aggregate risk since the total income of the economy is a constant and given by the value

$$\varpi = \sum_i \sum_j \mu_{ij} w(s_j) \lambda_i(s_i)$$

where with λ we denote the stationary probability distribution associated with μ. To get this result of no aggregate uncertainty we need to assume particular correlation between the individual shocks, to illustrate this consider the following example with an economy with a continuum of individuals characterized by a parameter i and by the realization of the random variable k uniformly distributed over the interval $[0, 1]$; endowment for each individual will take only two values (w_1, w_2) determined by the realization of k as follows: if $i \in \left[k, k + \frac{1}{2}\right]$ and if $k \leq \frac{1}{2}$ then the individual receives $w_1$and all others receive w_2; also if $i \in \left[k, k - \frac{1}{2}\right]$ if $k \geq \frac{1}{2}$ the individual receives w_2 as all the others receive w_1. We observe that for any value of i the individual receives w_1 and w_2 with probability $\frac{1}{2}$. If now we look at the aggregate endowment of the economy we have

$$\int_k^{k+\frac{1}{2}} w_1 di + \int_{k+\frac{1}{2}}^1 w_2 di + \int_0^k w_2 di = \frac{1}{2} w_1 + \frac{1}{2} w_2$$

we notice how this value is independent of the realization of the variable k . Going back to the original model we have the following house hold problem

$$\max \sum_t \sum_{s^t} \mu\left(s^t\right) \beta^t u\left(c_t\left(s^t\right)\right)$$

$$s.t. \quad \sum_t \sum_{s^t} q_t\left(s^t\right) \left[c_t\left(s^t\right) - w(s_t)\right] \leq 0$$

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A competitive equilibrium will be given by an allocation $\{c_t\}_{t=0}^{\infty}$ and a price system $\{q_t\}_{t=0}^{\infty}$ such that the above household problem is solved, the firm first order condition are satisfied and market clears, that is

$$\sum_{s^t} \mu\left(s^t\right) c_t\left(s^t\right) = \sum_{s^t} \mu\left(s^t\right) w\left(s_t\right), \quad \forall t. \tag{1}$$

If we consider a social planner problem for this economy we would have

$$\begin{aligned} &\max \sum_t \sum_{s^t} \beta^t \mu\left(s^t\right) u\left(c_t\left(s^t\right)\right) \\ &s.t. \quad \sum_{s^t} \mu\left(s^t\right) c_t\left(s^t\right) = \sum_{s^t} \mu\left(s^t\right) w\left(s_t\right), \quad \forall t \end{aligned}$$

Note how there exist a price system and allocations such that the two problems are equivalent, in particular take

$$\begin{aligned} q_t\left(s^t\right) &= \beta^t \mu\left(s^t\right), \\ c_t\left(s^t\right) &= \sum_{s^t} \mu\left(s^t\right) w\left(s_t\right) = \bar{w} \end{aligned}$$

Then we claim that the above allocations solve the planner problem, to see this from the planner problem FOC we get

$$\begin{aligned} \beta^t \mu\left(s^t\right) u'\left(c_t\left(s^t\right)\right) &= \lambda q_t\left(s^t\right) \\ &= \lambda \beta^t \mu\left(s^t\right) \end{aligned}$$

so that $u'\left(c_t\left(s^t\right)\right) = \lambda$ so we have constant values for the consumption levels. Notice now, how the only constant consumption stream that satisfies the household budget constraint is given by $\bar{c} = \frac{\bar{w}}{1-\beta}$.

We now write a recursive formulation for the above problem. We will allow households to accumulate assets $a\left(s_{t+1}|s_t\right)$, that determine the number of units of assets that pay one unit of consumption contingent on the state $s_{t+1}|s_t$. Our dynamic programming problem will be given by

$$\begin{aligned} v\left(a, s\right) &= \max \left\{ u\left(c\right) + \beta \sum_{s'} \mu\left(s|s'\right) v\left(a\left(s'|s\right), s'\right) \right\}, \qquad (2) \\ s.t. \qquad & c + \sum_{s'} q\left(s'|s\right) a\left(s'|s\right) \leq w\left(s\right) + a, \\ & a\left(s'|s\right) \geq -\phi. \end{aligned}$$

Our dynamic programming problem will be characterized by the following objects: $v\left(a, s\right) : \mathbb{R}^2 \to \mathbb{R}$ value function; $c\left(a, s\right) : \mathbb{R}^2 \to \mathbb{R}_+$, $g\left(a, s\right) : \mathbb{R}^2 \to \mathbb{R}_+^n$ policy functions; $q\left(s'|s\right) : \mathbb{R}^2 \to \mathbb{R}$ pricing kernel. So far we are considering our model in a complete market framework, where the households can insure against any possible realization of the random variable s, looking at the assets in this case we are assuming that the payoff matrix has full rank (INSERT EXAMPLE??). To characterize the stationary equilibrium we also introduce a probability distribution over the state space $\lambda\left(a, s\right)$ that will determine the fraction of households with given asset holding a in state s (note that we are also assuming that the set of possible asset holding is finite). A stationary recursive competitive equilibrium (stationary equilibrium in brief) is determined by a value function, policy functions, prices and equilibrium distribution such that

1. $v(a,s)$, $c(a,s)$, $g(a,s)$ solve the dynamic programming problem given in (**??**);

2. market clears

$$\sum_a \sum_s \lambda(a,s)\, c(a,s) = \sum_a \sum_s w(s)\, \lambda(a,s)$$

so that aggregate consumption equals aggregate output, we also impose market clearing for each individual asset

$$\sum_a \sum_s \lambda(a,s)\, a = 0 \quad \forall i = 1, \dots, n,$$

or

$$\sum_a \sum_s \lambda(a,s)\, g_i(a,s) = 0 \quad \forall i = 1, \dots, n,$$

where g_i is the i-th component of g, also

$$\sum_a \lambda(a,s) = \sum_{s_{-1}} \mu(s|s_{-1})\, \nu(s_{-1}),$$

where $\nu(s_{-1})$ is the invariant distribution associated with μ;

3. λ is a stationary distribution over the state space, that is

$$\lambda(a', s') = \sum_a \sum_s \lambda(a,s)\, I(a', s', a, s)\, \mu(s'|s),$$

where $I(a', s', a, s) = 1$ if $g(a,s) = a'$ and zero otherwise.

If now we consider s and a taking a continuum of values, then λ will be a stationary distribution over S if

$$\lambda_i([-\phi, \hat{a}]) = \int_{A_i(\hat{a})} \int_s \lambda_i(da)\, \mu(ds), \quad \forall i, \forall \hat{a} \in S.$$

where $\lambda_i([-\phi, \hat{a}])$ (the i-th component of λ) denotes the mass of people with asset holding of asset i in the interval $[-\phi, \hat{a}]$, and $A_i(\hat{a}) = \{a, s | g_i(a,s) \leq \hat{a}\}$; In this setting the market clearing condition has to be rewritten as

$$\int_a \int_s c(a,s)\, \lambda(da)\, \mu(ds) = \int_s w(s)\, \mu(ds).$$

The equilibrium will be characterized by the following prices and allocations

$$\begin{aligned} c &= \int w(s)\, \mu(ds) = \bar{w}, \\ q(s'|s) &= \beta \frac{\mu(s'|s)\, u'(c)}{u'(c)} = \beta u(s'|s). \end{aligned}$$

Assets holdings will be determined by the following relation $a + w(s) = \bar{w}$ for every value of s; note how each household will hold the same portfolio that is (assuming N households)

$$\begin{aligned} a_1 &= \bar{w} - w(1), \\ &\vdots \\ a_N &= \bar{w} - w(N), \end{aligned}$$

so that the stationary probability distribution will take values $\lambda_i(a_i) = 1$ if $a_i = a_1, \ldots, a_N$ and zero otherwise. Then we can rewrite the problem as

$$\begin{aligned} v(a,s) &= \max\left\{ u(c) + \beta \sum_{s'} \mu(s|s') \, v\left(a'(s'|s), s'\right) \right\}, \qquad (3) \\ s.t. & \qquad c + \sum_{s'} q(s'|s) \, a'(s'|s) \leq \bar{w}. \end{aligned}$$

That is a problem where all households start off with the same wealth and confront the same price. Then they will choose the same portfolio allocation, so that each household will choose complete insurance. Hence the equilibrium allocation is the same as in the planning problem. Note that this result depends heavily on the assumption that the state you observe is public information and that the probability of each state cannot be altered (household are price takers)

We now embed the above framework in a growth model. The uncertainty will be given by shock to the labor endowment $l(s)$; aggregate labor will be at a constant level given by $L = \sum_s \mu(s) \, l(s)$; also we introduce an aggregate production function given by $F(K, L)$. The dynamic programming problem faced by the households will be given by

$$\begin{aligned} v(a,s) &= \max\left\{ u(c) + \beta \sum_{s'} \mu(s'|s) \, v(a', s') \right\} \qquad (4) \\ s.t. & \qquad c + \sum_{s'} q(s'|s) \, a(s') \leq wl(s) + a(1+r). \\ & \qquad a(s'|s) \geq -\phi \qquad (5) \end{aligned}$$

Firms solve the following

$$\max\{F(K,L) - \hat{r}K - wL\}$$

with FOC given by

$$\begin{aligned} \hat{r} &= F_K(K,L), \\ w &= F_L(K,L) \end{aligned}$$

where $r = \hat{r} - \delta$ and δ the depreciation rate. A stationary recursive equilibrium will be a set of functions $v(a,s)$, $c(a,s)$, $g(a,s)$, $q(s'|s)$ a probability distribution $\lambda(a)$and real numbers w, r, K, L such that:

1. $v(a,s)$, $c(a,s)$, $g(a,s)$ solve the household problem;
2. λ is a stationary distribution;

3. w, r, δ solve the firm FOC;

4. $\int_a \int_s \lambda(da)\, \mu(s) = K$.

XXXXXInset Representative HH and 1st welfare considerationsXXXX

We now begin studying incomplete market models; In this framework we will consider a single asset available to households in order to smooth their consumption. We will modify the household problem described in equation (**??**) to the following

$$\begin{aligned} v(a,s) &= \max \left\{ u(c) + \beta \sum_{s'} \mu(s'|s)\, v(a', s') \right\} \qquad (6) \\ s.t. \quad & c + a' \leq wl(s) + a(1+r) \\ & a' \geq -\phi. \end{aligned}$$

as a technical assumption we set $l(s) > 0$, in other words we assume that there exists $B > 0$ such that $l(s) \geq B$ for all s. In order to assure that the household can always repay the debt we consider the natural debt limit of

$$\phi \leq \min_s \frac{wl(s)}{r}$$

We observe that the Lucas model can be viewed as a particular case of the above by setting $r = 0$ and $\phi = 0$.

We define a stationary equilibrium as a collection of functions $v(a,s)$, $c(a,s)$, $g(a,s)$ a probability distribution $\lambda(a)$ and a real number r, such that

1. $v(a,s)$, $c(a,s)$, $g(a,s)$ solve the household problem;

2. $\lambda([-\phi, \hat{a}]) = \int_{A_i(a')} \sum_s \lambda_i(da)\, \mu(s), \quad \forall i, \forall \hat{a} \in S$.

   (where $A(a') = \{a, s | g_i(a,s) \leq a'\}$)

3. $\int_a \sum_s c(a,s)\, d\lambda(a)\, \mu(s) = w \sum_s l(s)\, \mu(s)$

   we could also consider market clearing for the asset given by

   $$\int_a \sum_s g(a,s)\, d\lambda(a)\, \mu(s) = \int_a a d\lambda(a)\,(1+r) = 0$$

   but this condition would be redundant due to Walras law.

An algorithm to compute an equilibrium for this economy would proceed as follows:

1. guess a value for r;

2. guess a value for v_0 and perform value iteration;

3. determine the optimal policy functions after convergence of the value function;

4. determine the stationary distribution as a fixed point of condition 2 in the above definition;

5. check market clearing condition, if not satisfied return to step 1;

X XXX Missing consideration on market clearing condition XXXX

In order to characterize the equilibrium we could proceed in an alternate way by looking at the FOC and envelope condition, from (**??**) we have

$$\begin{aligned} u'(c(a,s)) &= \beta(1+r)\sum_{s}\mu(s')\,v_1(a',s') \\ v_1(a,s) &= u'(c(a,s)) \end{aligned}$$

so that

$$\begin{aligned} & u'(a(1+r)wl(s)-g(a,s)) = \\ = {} & \beta(1+r)\sum_{s}\mu(s')\,u'\big(g(a,s)(1+r)+wl(s')-g(g(a,s),s')\big) \end{aligned}$$

that gives us

$$u'(c_t) = \beta(1+r)\,E\left[u'(c_{t+1})\right] \tag{7}$$

suppose now $\beta(1+r)>1$ then $u'(c_t)>E[u'(c_{t+1})]$ this case will not lead to any equilibrium in fact we will also have $u'(c_{t+1})>E[u'(c_{t+2})]$ so that $E[u'(c_{t+2})]>E[u'(c_{t+3})]>\ldots.$
summarizing, we will have $u'(c_t)\to\infty$ as t goes to infinity, but then we would have $a_t\to\infty$ violating the condition on the natural debt limit

suppose $\beta(1+r)=1$ we will see that also in this case there cannot be any equilibrium (in contrast with the complete market case) suppose in fact there exist $a_{\max}$ such that $a\le a_{\max}$ in every time. We would then have (substituting envelope condition in (**??**)) that

$$\begin{aligned} v'(a_{\max}(1+r)+wl(\bar{s})) &= \beta(1+r)\,E\left[v'(a_{\max}(1+r)+wl(s))\right] \qquad (8) \\ &> v'(a_{\max}(1+r)+wl(\bar{s})) \end{aligned}$$

thus reaching a contradiction under the further assumption that v is concave and that l and g is increasing in s. By the previous discussion we then conclude that if an equilibrium exists we must have $\beta(1+r)<1$.

We now give a proof of existence of equilibrium in the incomplete market case. We will consider the household problem described in equation (**??**) in the case where s can take a continuum of values. Also note that if we considered the model proposed by Bewley-Aiyagari where the individual asset available to households is to be interpreted as physical capital, then condition 3 in the definition of stationary equilibrium should be replaced by the following

3' w and r are given by the firm FOC

$$\begin{aligned} w &= F_L\left(K,\int l(s)\,d\mu(s)\right) \\ r &= F_K\left(K,\int l(s)\,d\mu(s)\right)-\delta \end{aligned}$$

for market clearing condition

$$\int\int g\left(a,s\right)d\lambda\left(a\right)d\mu\left(s\right)=K,$$

together with

$$\int\int\left[c\left(a,s\right)+g\left(a,s\right)\right]d\lambda\left(a\right)d\mu\left(s\right)=F\left(K,\int l\left(s\right)d\mu\left(s\right)\right)+\left(1-\delta\right)K$$

Going back to the sketch of the proof we will have the following steps:

Step 1:

From the dynamic programming problem we derive $g\left(a,s;r\right)$ (which implicitly depends on r) that induces a Markov process for a.

Step2:

We want to show that the probability distribution determined by g converges to $\lambda\left(a,r\right)$, stationary distribution, as t goes to infinity that is

$$\lambda_t\left(a,r,a_0\right)\rightarrow\lambda\left(a,r\right)\quad\forall a_0,$$

we now want to verify if market clears

$$\int a_t d\lambda_t\left(a_t,r,a_0\right)\rightarrow\int a d\lambda\left(a,r\right)=0.$$

From equation (**??**) define $M_t=\beta^t\left(1+r\right)u'\left(c_t\right)$ so that we rewrite (**??**) as

$$E\left[M_{t+1}-M_t\right]\leq0$$

we observe that M_t is a supermartingale. By the supermartingale convergence theorem if the above holds M_{t+1} converges in distribution to a non negative random variable: $M_t\rightarrow\hat{\lambda}\left(M\right)$. This implies that c_t converges, so that $a_t\rightarrow\lambda\left(a\right)$.

Step 3:

By theorem 12.13 of SLP we obtain λ continuous in r given $g\left(a,s;r\right)$ is continuous in r. We now call $E\left[a|r\right]=\int a d\lambda\left(a,r\right)$; the next step is to show that $E\left[a|r\right]<0$ for small values of r and that $E\left[a|r\right]>0$ for large values of r.

Step 4:

We show that $E\left[a|r\right]<0$ for r small; consider $r=-1$ then by looking a the dynamic programming problem in (**??**) we observe that $a'=-\phi$ since there will be no incentive in repaying the debt.

Also we have $E\left[a|r\right]>0$ for r large; to see this consider a value of r such that $\beta\left(1+r\right)$ then, as we have seen before $c_t\rightarrow\infty$ and also $a_t\rightarrow\infty$.

We can now conclude the proof by using intermediate value theorem, that is there exist a value of r such that $E\left[a|r\right]=0$. Note that this does not imply uniqueness, so we could have multiple equilibria.

We conclude by characterizing the equilibrium. From (**??**) we consider the i.i.d. case

$$\begin{aligned} & v(a,s) = \max\left\{u(c) + \beta \int_{s'} v\left(a', s'\right) d\mu\left(s'\right)\right\} \qquad (9) \\ s.t. \qquad & c + a' \leq wl(s) + a(1+r) \\ & a' \geq -\phi. \end{aligned}$$

we now introduce a new variable, defined as the disposable wealth, $z = wl(s) + a(1+r) + \phi$. We can then rewrite (**??**) as

$$\begin{aligned} v(z) &= \max\left\{u\left(z - a' - \phi\right) + \beta \int_{s'} v\left(a' + \phi + ra' + wl(s)\right) d\mu\left(s'\right)\right\} \\ s.t. \qquad & c + a' + \phi \leq z \\ & z' = wl\left(s'\right) + a'(1+r) + \phi \end{aligned}$$

note that we do not consider anymore s as a state variable.

We now show that if $\beta(1+r) = 1$ then $g(z) > z$ for all z so that there cannot be any equilibrium; suppose not, then exists a value of the wealth $z_{\max}$ such that $g(z_{\max}) = z_{\max}$; following the same procedure as for equation (**??**) from first order and envelope condition we obtain

$$\begin{aligned} v'(z_{\max}) &= E\left[v'(z_{\max} + \phi + rz_{\max} + wl(s))\right] \\ &> v'(z_{\max} + \phi + rz_{\max} + wl(\bar{s})) \\ &> v'(z_{\max}) \end{aligned}$$

As before in order to obtain an equilibrium, we must have $\beta(1+r) < 1$. Recall that in the complete market framework $\beta(1+r) = 1$ was an acceptable value for equilibrium, thus in Aiyagary and similar incomplete market models we would obtain an equilibrium value of r that is lower than the analogous in the complete market case, this will also imply that the levels of capital stock will be strictly bigger in presence of incomplete markets, this is so because you can't have insurance for all possible states, thus household tend to accumulate k.