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Lecture 1*

Idiosyncratic Risk affecting production decisions

We studied some models with idiosyncratic shocks in the consumer's side, now we will see a model by Hopenhayn[1] which consider shocks to the firms. This model is built in the tradition of the Marshallian model of enter and exit, which has as its main feature the possibility of firms entering and exiting the market in equilibrium.

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Structure of the model

The industry is described by a continuum of competitive firms producing a homogeneous good. The aggregate demand is given by the inverse function $D(Q)$ and the price of the only input used is given by $W(N)$, where N is the total input supply.

Assumptions:

A.1: **(a)** D is continuos, strictly decreasing and $\lim_{x\to\infty} D(x) = 0$; **(b)** W is continuous, nondecreasing and strictly bounded away from zero.

Let $q = f(\varphi, n)$ be the product of each firm and $\varphi \in S = [0, 1]$ represents the individual productivity shock which evolves according to a Markov process independent across firms with conditional distribution $F(\varphi\prime/\varphi)$. Each firm in the market must pay every period a fixed cost c_f. For given prices p and w, let $\pi(\varphi, p, w)$ be the profit function[2], $q(\varphi, p, w)$ the output supply and $n(\varphi, p, w)$ the input demand functions.

A.2: **(a)** q and n are single-valued, strictly increasing in φ and continuous; **(b)** π is continuous and strictly increasing in φ; **(c)** $\lim_{Q\to 0} \pi(0, D(Q), w) > 0$.

A.3: **(a)** F is continuous in φ and $\varphi\prime$; **(b)** F is strictly decreasing in φ.

A.4: For all $\varepsilon > 0$ $\exists$ $N \in \mathbb{Z}$ such that $F^n(\varepsilon/\varphi) > 0$, where $F^n(./\varphi)$ denotes the distribution of φ_{t+n} given φ_t

By assumption A.4 every firm has always a positive chance of getting a very low φ, then it implies that the expected lifetime of a firm is almost surely finite, given that it will decide to leave the market if its productivity shock goes below a cutoff level.

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[1]See Hopenhayn, H. (1992): "Entry, Exit and Firm Dynamics in Long Run Equilibrium", *Econometrica,* 60, 1127-1150.

[2]I.e., $\pi(\varphi, p, w) = \max_n pf(\varphi, n) - wn - c_f$

The characterization of the industry is such that in each period, incumbent firms can leave the market and new can enter. Each entrant has to pay an entry cost $c_e \geq 0$ and draws a productivity shock from an initial distribution υ.

A.5: υ has a continuous distribution function G.

The timing of the model is the following: after observing their shocks, the firms decide optimally the output level and the prices are determined competitively to equate supply and demand. We denote μ_t the measure of firms by productivity shocks, i.e., $\mu(A)$ is the mass of firms with shock in A, for all Borel set $A \subseteq S$. The measure $M_t = \mu_t(S)$ gives the total size of the industry.

The aggregate variables of the economy, respectively, the aggregate output supply and aggregate input demand functions, are given by:

$$Q^s(\mu, p, w) = \int q(\varphi, p, w)\mu(d\varphi), \tag{1}$$

$$N^d(\mu, p, w) = \int n(\varphi, p, w)\mu(d\varphi). \tag{2}$$

Now we are ready to characterize the firm's problem. Since each firm can choose to exit the market, we have to define the maximization problem for the incumbent firms and for the entrants.

Given the sequence of prices $z = \{p_t, w_t\}$, the problem for an incumbent firm is given by:

$$v_t(\varphi, z) = \pi(\varphi, p, w) + \beta \max\{0, \int v_{t+1}(\varphi', z)F(d\varphi'/\varphi)\}. \tag{3}$$

The second term in this Bellman equation reflects the firm's choice of exiting the market or staying. Note that this decision has to be taken before the next period shock is know. The assumptions A.2 and A.3 imply that the optimal decision of the firm will be a reservation rule $x_t(z)$, i.e., if $\varphi \geq x_t(z)$ the firm stays in the market and it exits otherwise. More formally, $x_t(z)$ is given by:

$$x_t = \inf\{\varphi \in S : \int v_{t+1}(\varphi', z)F(d\varphi'/\varphi) \geq 0\} \tag{4}$$

$$x_t = 1 \text{ if this set is empty.}$$

The problem of an entrant firm is the maximization of its discounted profit, given by:

$$v_t^e(z) = \int v_t(\varphi, z)\nu(d\varphi). \tag{5}$$

If we denote M_t the mass of entrants in period t, we have that the free entry condition implies that $v_t^e(z) \leq c_e$, with equality if $M_t > 0$.

We can see that the exit and entry optimal rules determines the evolution of the state of the industry μ_t, given by:

$$\mu_{t+1}([0,\varphi']) = \int_{\varphi \geq x_t} F(\varphi'/\varphi)\mu_t(d\varphi) + M_{t+1}G(\varphi'), \tag{6}$$

for all $\varphi' \in S$.

Equation (**??**) gives the probability of having, in period $t+1$, firms with shock in the interval $[0,\varphi']$, which can happen if the firm stay in the market and get a productivity shock in this interval (the first term) or if the firm enters the market (second term).

Alternatively we can define:

$$\widehat{P}_t(\varphi, A) = \begin{cases} \int_A F(d\varphi'/\varphi) & \text{if} \quad \varphi \geq x_t \\ 0 & \text{otherwise} \end{cases}$$

Then (**??**) can be written as:

$$\mu_{t+1} = \widehat{P}_t\mu_t + M_{t+1}\nu. \tag{7}$$

For a better understanding of these definitions, consider a simple example with only two possible values for the productivity shock, with transition probabilities $f^{ij} = prob(\ \varphi' = j/\varphi = i)$ and let ν_i denote the initial probability of having the shock i. Suppose that the optimal decision rule is such that if $i = 1$ exit and if $i = 2$ stay. Then:

$$\mu_{t+1} = \begin{bmatrix} \mu_{t+1}^1 \\ \mu_{t+1}^2 \end{bmatrix} = \begin{bmatrix} 0 & f^{12} \\ 0 & f^{22} \end{bmatrix} \begin{bmatrix} \mu_t^1 \\ \mu_t^2 \end{bmatrix} + M_{t+1} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}$$

We are now ready to define a competitive equilibrium for this environment.

A competitive equilibrium is the sequences of functions $\{p_t, w_t, Q_t, N_t, M_t, x_t, \mu_t\}_{t=0}^{\infty}$ such that:

(i) $p_t = D(Q_t)$
  $w_t = W(N_t)$
  $Q_t = \int q_t(\varphi, p, w)\mu_t(d\varphi)$
  $N_t = \int n_t(\varphi, p, w)\mu_t(d\varphi)$

(ii) x_t satisfies (4)

(iii) $v_t^e(z) \leq c_e$ with equality if $M_t > 0$
  μ_t is defined recursively by (**??**) given μ_0, M_t and x_t, where μ_0 is the initial distribution over firms' shocks.

In Hopenhayn's paper you can find proofs of existence and uniqueness of the competitive equilibrium.

An important feature of this problem is that, since the only uncertainty is the firms' shocks and since there are a large number of firms (the law of large numbers holds) and the conditional distribution F, as well as the probability over the initial state υ, are the same for all firms. This implies that the frequency of the distribution of the shocks will be the same than the one implied by the conditional distribution and the entry and exit rules. This result implies that the aggregate output, employment, prices and the frequency distribution of φ are not stochastic. In other words, given μ_0, the C.E. prices and μ_t are deterministic and for the firms' perspective, the only stochastic variable is its productivity shock.

Now we focus on a stationary equilibrium in this environment. A stationary equilibrium is defined as a vector $\{p, w, Q, N, x, M, \mu\}$ such that $p = p_t, w = w_t, Q = Q_t, N = N_t, x = x_t, M = M_t, \mu = \mu_t$ and $\{p_t, w_t, Q_t, N_t, M_t, x_t, \mu_t\}$ is an equilibrium for $\mu_0 = \mu$. To address the existence of a stationary equilibrium let's consider a finite set for the possible values of the productivity shocks, $S = \{\varphi_1, ..., \varphi_N\}$, with transition probabilities f^{ij} and distribution $\mu = \{\mu_1, ..., \mu_N\}$. It can be shown that for any distribution μ, there exist a unique vector of aggregate variables and prices such that (i) is satisfied in the definition of competitive equilibrium. Then, given μ, we have:

$$D(\sum_{j=1}^{N} \mu_j q(\varphi_j, p, w)) = p,$$

$$W(\sum_{j=1}^{N} \mu_j n(\varphi_j, p, w)) = w.$$

For each $i \in \{1, .., N\}$, the firms' problem is given by:

$$v(\varphi_i, p(\mu), w(\mu)) = \pi(\varphi_i, p(\mu), w(\mu)) + \beta \max\{0, \sum_{j=1}^{N} f^{ij} v(\varphi_j, p(\mu), w(\mu))\}$$

¿From the properties of π and assumption A.3, can be shown that there exist a unique v that solves this problem which is increasing in φ, thus exists a $x(\mu)$ that gives the exit rule [3]. Then, in a stationary equilibrium we would have, as defined in (**??**)

$$\mu = P_X \mu + M\nu \Rightarrow \mu = M\nu[I - P_X].$$

A condition to have an equilibrium with positive entry, that is $M > 0$, is that:

$$\sum_{j=1}^{N} v(\varphi_j, p(\mu), w(\mu))\nu_j = c_e$$

[3]The results are proved in Hopenhayn (1992) for the case where $S = [0, 1]$.

On the other hand, if we have that $\sum_{j=1}^{N} v(\varphi_j, p(\mu), w(\mu))\nu_j < c_e$, we would have an equilibrium with zero entry.

After showing that a stationary equilibrium exists, the model can be used to characterize some properties of this equilibrium, assuming specific conditions on the distributions (F, ν). One possible application is to discuss the life cycle of a firm, i.e., the model can interpreted as one that generates a distribution of the size of the firms by age. In the paper, the author shows conditions on μ_t, the distribution of shocks for firms with age t ($\mu_1 = \nu; \mu_{t+1} = P_x \mu_t$), is increasing in t. Then the model would reproduce an empirical behavior that the size of a firm is increasing in age.