

Lecture 5\*

Models of Overlapping Generations<sup>1</sup>

Consider a two period OLG model. At each time  $t$ , a generation is born and lives for two period and then dies. Any generation is called “young” at first period of life and called “old” at the second period. Allocation of the generation  $t$  is denoted by subscript “1” when they are young and “2” when they are old. Each generation is endowed with  $(w_{1t}, w_{2t+1})$  and has preferences  $U(c_{1t}, c_{2t+1})$  over consumption in young period and old period. Population is assumed to be constant (normalized to one).

At date zero, there is an initial old (born at date -1), that lives only one period and has endowment  $w_{20}$  and has preference  $U^0(c_{20})$ .

**Definition 1** A competitive equilibrium is allocations  $\{(c_{1t}, c_{2t+1})\}_{t \geq 0}$ ,  $c_{20}$  and prices  $\{p_t\}_{t \geq 0}$ , such that

i. Given prices,  $\{(c_{1t}, c_{2t+1})\}_{t \geq 0}$  solve household's problem for all  $t$ :

$$\begin{aligned} & \max_{(c_{1t}, c_{2t+1})} U(c_{1t}, c_{2t+1}) \\ & \text{s.t.} \\ & p_t c_{1t} + p_{t+1} c_{2t+1} \leq p_t w_{1t} + p_{t+1} w_{2t+1} \\ & c_{1t} \geq 0 \\ & c_{2t} \geq 0 \end{aligned}$$

and,

Given  $p_0$ ,  $c_{20}$  solves initial old household's problem:

$$\begin{aligned} & \max_{c_{20}} U^0(c_{20}) \\ & \text{s.t.} \\ & p_0 c_{20} + \leq p_0 w_{20} \\ & c_{20} \geq 0 \end{aligned}$$

ii. Markets are clear for all  $t$ :

$$c_{1t} + c_{2t} = w_{1t} + w_{2t}$$

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\*These notes are prepared by Laurence Ales, Roozbeh Hosseini, Priscila Maziero and Miguel Ricaurte. They are preliminary and possibly contain errors. Comments and feedbacks are welcome.

<sup>1</sup>See Ljungqvist-Sargent(2001), chapter 8. Also Krueger(2002), chapter 8.

**Example 1:** Let  $U(c_1, c_2) = u(c_1) + \beta u(c_2)$  and  $U^0(c_2) = \beta u(c_2)$  ( $u(\cdot)$  strictly increasing), and  $(w_1, w_2) = (0, 1)$ . Let  $p_0 = 1$ , then initial old optimal choice is  $c_{20} = 1$ , and therefore market clearing implies

$$(c_{1t}, c_{2t}) = (0, 1) \quad \text{for all } t$$

And from first order conditions

$$\frac{p_{t+1}}{p_t} = \frac{\beta u'(c_{2t+1})}{u'(c_{1t})} = \frac{\beta u'(1)}{u'(0)}$$

**Example 2:** Consider the same preferences as example 1, but with endowment  $(w_1, w_2) = (1, 0)$ . Then

$$(c_{1t}, c_{2t}) = (1, 0) \quad \text{for all } t$$

and

$$\frac{p_{t+1}}{p_t} = \frac{\beta u'(c_{2t+1})}{u'(c_{1t})} = \frac{\beta u'(0)}{u'(1)}$$

Now let  $\frac{\beta u'(0)}{u'(1)} = 1$ , therefore  $p_t = 1$  for all  $t$ .

### Welfare Analysis

Because in this economy there are infinitely many agents, the total value of endowment is not in general finite and standard proof of first welfare theorem fails. There are non-efficient competitive equilibria in these models. In particular, competitive equilibrium allocation in example 2 (with the above assumption) is not pareto optimal.<sup>2</sup>

### Money in OLG Models

We introduce fiat money by endowing initial old with stock of  $M$ .

**Definition 2** *Monetary Equilibrium* A competitive equilibrium is allocations  $\{(c_{1t}, c_{2t+1})\}_{t \geq 0}$ ,  $c_{20}$  and prices  $\{p_t\}_{t \geq 0}$  and  $q$  such that

i. Given prices,  $\{(c_{1t}, c_{2t+1})\}_{t \geq 0}$  solve household's problem for all  $t$ :

$$\begin{aligned} & \max_{(c_{1t}, c_{2t+1})} U(c_{1t}, c_{2t+1}) \\ & \text{s.t.} \\ & p_t c_{1t} + p_{t+1} c_{2t+1} \leq p_t w_{1t} + p_{t+1} w_{2t+1} \\ & c_{1t} \geq 0 \\ & c_{2t} \geq 0 \end{aligned}$$

and,

Given  $p_0$  and  $q_0$ ,  $c_{20}$  solves initial old household's problem:

$$\begin{aligned} & \max_{c_{20}} U^0(c_{20}) \\ & \text{s.t.} \\ & p_0 c_{20} + \leq p_0 w_{20} + q_0 M \\ & c_{20} \geq 0 \end{aligned}$$

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<sup>2</sup>See Krueger(2002), for more discussion on this

ii. Markets are clear for all  $t$ :

$$\begin{aligned} c_{1t} + c_{2t} &= w_{1t} + w_{2t} \\ q_t M &= p_t(w_{1t} - c_{1t}) \end{aligned}$$

A Monetary Equilibrium is a Competitive Equilibrium with  $q_t > 0$  for all  $t$ .

**Theorem 1** A Non-Monetary Equilibrium (where  $q_t = 0$  for all  $t$ ) always exists.

**Theorem 2** A Monetary Equilibrium exists iff Non-Monetary equilibria are inefficient

In what follows assume that endowment process is stationary,  $(w_{1t}, w_{2t+1}) = (w_1, w_2)$ .

### Alternative formulation of household problem

When  $M > 0$ , we can write the household problem in the following alternative form

$$\begin{aligned} \max_{(c_{1t}, c_{2t+1})} & U(c_{1t}, c_{2t+1}) \\ \text{s.t.} & \\ p_t c_{1t} + qM & \leq p_t w_{1t} \\ p_{t+1} c_{2t+1} & \leq p_{t+1} w_{2t+1} + qM \\ c_{1t} & \geq 0 \\ c_{2t} & \geq 0 \end{aligned}$$

**Theorem 3** In a two period OLG model, Monetary Exists iff  $\frac{U_1(w_1, w_2)}{U_2(w_1, w_2)} < 1$ . And autarky is efficient iff  $\frac{U_1(w_1, w_2)}{U_2(w_1, w_2)} = 1$ .

**proof:** Suppose  $\frac{U_1(w_1, w_2)}{U_2(w_1, w_2)} \geq 1$  and a monetary equilibrium exists, i.e.  $q > 0$ . From budget constraint

$$\begin{aligned} c_{1t} &= w_1 - \frac{q}{p_t} M < w_1 \\ c_{2t+1} &= w_2 - \frac{q}{p_{t+1}} M > w_2 \end{aligned}$$

From first order conditions

$$\frac{p_t}{p_{t+1}} = \frac{U_1(c_{1t}, c_{2t+1})}{U_2(c_{1t}, c_{2t+1})} > \frac{U_1(w_1, w_2)}{U_2(w_1, w_2)} \geq 1$$

Now let  $\theta = \frac{U_1(w_1, w_2)}{U_2(w_1, w_2)}$ . If  $\theta > 1$ , then

$$p_t > \theta p_{t+1}$$

Which implies that  $p_t$  converges to zero. But from (money) market clearing we have

$$qM = p_t(w_{1t} - c_{1t})$$

This is a contradiction since we assumed  $q > 0$ .

If  $\theta = 1$ , then  $p_t$  converges to some  $\bar{p} \geq 0$ . If  $\bar{p} = 0$  the argument above shows the contradiction. If  $\bar{p} > 0$ , then  $\frac{p_t}{p_{t+1}}$  converges to 1. Then FOC implies

$$\frac{U_1(c_{1t}, c_{2t+1})}{U_2(c_{1t}, c_{2t+1})} \rightarrow \frac{U_1(w_1, w_2)}{U_2(w_1, w_2)} = 1$$

And this together with (goods) market clearing implies

$$\begin{aligned} c_{1t} &\rightarrow w_1 \\ c_{2t+1} &\rightarrow w_2 \end{aligned}$$

therefore,

$$qM = p_t(w_1 - c_{1t}) \rightarrow 0$$

contradiction.

### **Non-Stationary Monetary Equilibrium**

[This part will be added later, read Ljungvist-Sargent Chapter 8]