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Comparing competition and collusion: a numerical approach[*]

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Summary. Collusion is a serious problem in many procurement auctions. In this research, I study a model of first price sealed bid procurement auctions with asymmetric bidders. I demonstrate that the equilibrium to the model is unique and describe three algorithms that can be used to compute the inverse equilibrium bid functions. I then use the computational algorithms to compare competitive and collusive bidding. The algorithms are useful for structural estimation of auction models and for assessing the damages from bid-rigging.

Keywords and Phrases: Asymmetric auction, Collusion.

JEL Classification Numbers: C72, L41.

1 Introduction

Bid-rigging is a serious problem in many procurement auctions. According to Pesendorfer (1995), bid-rigging accounts for 50 percent of the cases filed by the Justice Department's anti-trust division that result in a criminal conviction. In cities with a long-standing relationship between organized crime and unions, such as New York and Chicago, bid-rigging has been and may still be a serious impediment to competition and efficiency in many industries.

There have been many instances of bid-rigging in the construction industry. According to Engineering News-Record, criminal bid-rigging cases have recently been filed in New York and Chicago for school construction, bridge repair, interior remodeling, paving and many other types of construction. In the New York

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cement industry in the 1980's, organized crime turncoats alleged that the Mafia designed an elaborate bid-rigging scheme that inflated building costs, making, for instance, the price of poured concrete the highest in the nation.[1]

Despite the prevalence of bid rigging, according to anti-trust officials, economic theory and econometrics play at best a minor role in the detection of collusive bidding. This is at first surprising, especially given the large body of academic work on cartel behavior in auctions and collusion more generally.

An important step to understanding collusion is building more realistic models of bidding. In Bajari (1997) and Bajari and Ye (2000), I argue that realistic models of bidding for procurement contracts should have asymmetric bidders. One important source of asymmetries is location. For instance, in Bajari (1997), I study the bidding by highway construction firms and find that 75 percent of the contracts in the data set are won by the closest firm. Porter (1999) in his study of bidding by dairies for contracts to supply school milk also finds that location plays a central role in firms' bidding strategies. Asymmetries may also occur because some firms are better managed than others, because collusion is present in the market or a variety of other reasons. In fact, it is hard to think of any real world procurement auction where firms are ex ante identical.

The equilibrium in the asymmetric auction model is technically difficult to analyze. The inverse bid functions can be characterized by a set of ordinary differential equations that are singular at the boundary conditions. Therefore, this system of differential equations does not satisfy a Lipschitz condition which is required in standard methods for proving the uniqueness of the solution to initial value problems. A Lipschitz condition is also required in many algorithms for computing solutions to initial value problems (see Gear, 1971).

This research overcomes two important technical obstacles to studying the asymmetric auction model. First, this research proves the uniqueness of equilibrium. Recently, Maskin and Riley (1996b) and LeBrun (1999) have also demonstrated the uniqueness of equilibrium in the N asymmetric bidder case.[2]

Second, this research describes three computational algorithms used in Bajari (1997) and Bajari and Ye (2000) to compute the equilibrium to the asymmetric auction model. Marshall et al. (1994) compute solutions to asymmetric auction models with 2 bidders. Riley and Li (1997) compute the equilibrium to asymmetric auction models, but I have found in practice, their algorithm is not sufficiently fast for econometric applications. Froeb et al. (1998) develop an algorithm to compute bid functions for N asymmetric bidders when valuations are distributed extreme value. Athey (1997) has computationally studied the solution to some common value auctions and Armential, Florens and Richard (1997)

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[1] In his biography, Mafia turncoat Sammy (The Bull) Gravano, a former member of the "Concrete Club" stated, "If one of them (contractor) gets a contract for, say, thirteen million, the next thing you know, after he knows he's got it, he jacks up the whole thing before it's over to a sixteen-or seventeen-million-dollar job. Now he's increased the cost thirty-three percent. So our greed (the Mafia) is compounded by the greed of them so-called legitimate guys (contractors)." Maas (1997 p. 271).

[2] The method of proof in this paper is a revision of Bajari (1997) and was derived independently of Maskin and Riley (1996b) and LeBrun (1999). The proof also relies on a different strategy.

have proposed the concept of constrained equilibrium to study bidding in auction models. Source code for these algorithms can be found on my web page.[3]

Third, this research studies the comparative statics of competitive and collusive bidding. I study the comparative statics of a firm's markup, profitability and probability of winning as the number of bidders changes and as cost parameters change. Also, this research compares a simple model of collusive bidding to a model of competitive bidding. I find some differences between competitive and collusive bidding that may be used in the empirical detection of collusion. The methods developed in this paper can be used to simulate the damages from collusion.

The paper has four major sections. First I develop a simple model of bidding with asymmetric firms. Second, I sketch a proof of the uniqueness of equilibrium and three methods for computing equilibrium are described. Third, the comparative statics of the bid functions are studied computationally. Fourth, technical details justifying the uniqueness proof and the computations are contained in an appendix.

2 A simple model of a procurement auction with asymmetric bidders

This section develops a simple model of a procurement auction with asymmetric bidders. The firms submit sealed bids for a contract to complete a single and indivisible project. In the model, firms have private values and are risk neutral. Unlike most auction models, we allow firms to be asymmetric in the sense that we do not require each firm's private information to be ex ante identically distributed.

The model is developed in three steps. First, the information structure of the model is described. Next, the vNM utility function and expected profit function for the firms are defined. Lastly, the equilibrium bidding functions are characterized.

2.1 Information

In the model, N firms compete for a contract to build a project. Before bidding starts, each firm i forms an estimate of its cost to complete the project. The cost estimate is firm i's private information, that is, firm i knows its own cost estimate but does not know of the cost estimates of other firms. The cost estimate for firm i is a random variable C_i with a realization denoted as c_i and is drawn independently across all firms. The random variable C_i has a cumulative distribution function $F_i(\cdot;\theta_i)$ and probability density function $f_i(\cdot;\theta_i)$ where θ_i is a vector of parameters. We will let $\theta = (\theta_1, ..., \theta_N)$ denote the vector of all firm specific parameters. The cost distribution is assumed to have support $[\underline{c}, \overline{c}]$ for all firms.

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[3] The url is http://www.stanford.edu/~bajari

As a simple example, consider a situation where each firm has a different location and hence different transportation costs. Then one natural specification for firm i's private information is:

$$c_i = constant + \beta_1 * distance_i + \varepsilon_i \tag{1}$$

In equation (1), c_i, firm i's cost estimate is a function of three terms. The first term is a constant, which could reflect attributes of the project that affect all firms identically, such as how many miles of highway must be paved or the tons of concrete that must be poured for the foundation of a new building. The second term, $\beta_1 * distance_i$, is different for each firm since each firm has a unique location. The third term, ε_i, is an independent random variable, which serves to model private information about some component of firm i's cost, such as the cost for materials or labor. If ε_i is normally distributed with mean zero and standard deviation σ then $\theta_i = (constant, \beta_1, distance_i, \sigma)$.

2.2 The vNM utility function

Let b_i denote the bid submitted by firm i. If firm i submits the lowest bid then firm i's vNM utility is $b_i - c_i$, and if it fails to submit the lowest bid then firm i's utility is assumed to be 0. If two or more firms submit the same bid, the contract will be awarded at random among the set of low bidders. But ties will never occur with positive probability in equilibrium. Firm i's vNM utility function can be written as:

$$u_i(b_1, ..., b_n, c_i) = \begin{cases} b_i - c_i & \text{if } b_i < b_j \text{ for all } i \neq j \\ & 0 \text{ otherwise.} \end{cases} \tag{2}$$

Firm i is said to have private values since its utility depends only on c_i and not the private information of other firms. Bajari (1997,2000), Armantier, Florens and Richard (1997) and Porter and Zona (1993) all use the assumption of private values in their models of procurement auctions. If firm specific factors account for the differences in cost estimates, then the assumption of private values is plausible. Often, labor and material costs are firm specific.

2.3 Expected profit

In the model, firm i's strategy is a function $b_i = \mathbf{b_i}(\mathbf{c_i}; \theta)$ which maps firm i's cost draw, c_i, to a bid b_i in the interval $[\underline{c}, \overline{c}]$. LeBrun (1994) and Maskin and Riley (1996a,b) have shown that, in equilibrium, the bid functions $b_i = \mathbf{b_i}(\mathbf{c_i}; \theta_\mathbf{i})$ are strictly increasing and differentiable which implies that the inverse bid functions $\phi_i(b; \theta) = \mathbf{b_i^{-1}}(\mathbf{b}; \theta)$ are also strictly increasing and differentiable. To simplify the notation, I will often write the bid function as $\mathbf{b_i}(\mathbf{c_i})$, suppressing its dependence on the vector of parameters θ.

In order to win the contract, firm i must submit the low bid. If firm i submits a bid of b_i, it will win the contract when $c_j \geq \phi_j(b_i)$ for all $j \neq i$, that is, all firms $j \neq i$ have cost draws greater than $\phi_j(b_i)$. Firm i's expected profit from bidding b_i when firm i has estimated cost of c_i will be denoted by $\pi_i(b_i, c_i; \theta)$. The expected profit satisfies:

$$\pi_i(b_i, c_i; \mathbf{b_{-i}}, \theta) = (\mathbf{b_i} - \mathbf{c_i})\mathbf{Q_i}(\mathbf{b}; \theta) \tag{3}$$

where $Q_i(b;\theta) = \prod_{j \neq i} 1 - F_j(\phi_j(b;\theta);\theta_j)$ is the probability that firm i is the lowest bidder. As we can see from equation (3), firm i's expected profit is a markup times the probability that firm i is the low bidder.

2.4 Properties of equilibrium

This section summarizes some well-known properties of the asymmetric auction model. First, I define an equilibrium in pure strategies.

Definition. An **Equilibrium in Pure Strategies** is a collection of functions $\mathbf{b_1^*}, \ldots, \mathbf{b_N^*}$ such that for all i, and for all $c_i \in [\underline{c}, \overline{c}]$, $b_i = \mathbf{b_i^*}(\mathbf{c_i})$ maximizes $\pi_i(\cdot, c_i; \mathbf{b_{-i}^*}, \theta)$.

The first order condition for maximizing firm i's expected profit is:

$$\frac{\partial}{\partial b_i}\pi_i(b_i, c_i; \theta) = (b_i - c_i)Q_i'(b_i;\theta) + Q(b_i;\theta) = 0 \tag{4}$$

In equilibrium, holding the bid functions of all the other firms fixed, the marginal benefit to firm i of increasing her bid must be equal to the marginal cost. The marginal cost is equal to $(b_i - c_i)Q_i'(b_i;\theta)$. This is the expected payment times the decrease in the probability of winning when firm i submits a higher bid. The marginal benefit to firm i of increasing her bid is $Q_i(b_i;\theta)$. This is the expected increase in her profit from submitting a higher bid.

Next, I show that the equilibrium to the model can be characterized as the solution to a system of differential equations with boundary conditions. Rearranging equation (4):

$$\begin{aligned}
\frac{\partial}{\partial b_i}\pi_i(b_i, c_i;\theta) &= (b_i - c_i)Q_i'(b_i;\theta) + Q(b_i;\theta) = 0, \quad i = 1, \ldots, N \\
\frac{\partial}{\partial b_i}\pi_i(b_i, c_i;\theta) &= \prod_{j \neq i}[1 - F_j(\phi_j(b_i))] - (b_i - c_i)\left\{\sum_{j \neq i} f_j(\phi_j(b_i))\phi_j'(b_i)\right. \qquad (5) \\
&\quad \left. \times \prod_{k \neq j}[1 - F_k(\phi_k(b_i))]\right\} = 0, \quad i = 1, \ldots, N.
\end{aligned}$$

Collecting terms and rewriting the system in terms of $\phi_j'(b)$:

$$\phi_i'(b) = \frac{1 - F_i(\phi_i(b))}{(N-1)f_i(\phi_i(b))}\left[\frac{-(N-2)}{b_i - \phi_i(b)} + \sum_{j \neq i} \frac{1}{b - \phi_j(b)}\right], \quad i = 1, \ldots, N. \tag{6}$$

Equation (6) shows that the inverse of the bid functions can be characterized as a system of N ordinary differential equations.

We now restate two key results from the theory literature that guarantee the existence of equilibrium and characterize the inverse bid functions. First, however, we state two regularity conditions.

Assumption 1 For all i, $F_i(c_i;\theta)$ has support $[\underline{c}, \overline{c}]$. The probability density function $f_i(c_i;\theta)$ is continuously differentiable.

Assumption 2 For all i, $f_i(c_i;\theta)$ is bounded away from zero on $[\underline{c}, \overline{c}]$.

Theorem 1 *(LeBrun (1994, 1995a,b) and Maskin and Riley (1996a,b)). If Assumptions 1 and 2 hold, then an equilibrium in pure strategies exists. Furthermore, the equilibrium bid functions are strictly increasing and differentiable.*

Another basic result from the theory literature is that the inverse bid functions can be characterized as the solution to a system of N differential equations with $2N$ boundary conditions.

Theorem 2 *(LeBrun (1995a), Maskin and Riley (1996a)). Suppose that Assumptions 1 and 2 hold. Let $\phi_1^*(b), \ldots, \phi_N^*(b)$ be inverse equilibrium bidding strategies. Then*

(i) For all i, $\phi_i^*(\overline{c}) = \overline{c}$,

(ii) There exists a constant β such that for all i, $\phi_i^*(\beta) = \underline{c}$,

(iii) For all i and for all $b \in [\beta, \overline{c}]$, equation (6) holds.

Conditions (i) and (ii) of Theorem 2 give $2N$ boundary conditions to the set of N differential equations in (6). Condition (i) says that bidders who have the highest possible cost draw $\overline{c}$ will also bid $\overline{c}$. The second condition (ii) says that at the lowest possible cost draw $\underline{c}$, all firms will bid β.

3 Uniqueness of equilibrium and computation

In this section, I outline a proof of the uniqueness of the equilibrium and describe three algorithms to compute the inverse equilibrium bid functions. Standard proofs of the uniqueness of the solution to differential equations with boundary conditions require that the system is Lipschitz. The Lipschitz condition does not hold for equation (6) at the boundary condition (i), $\phi_i^*(\overline{c}) = \overline{c}$ for all i. At the boundary condition (i), $1 - F_i(\phi_i^*(\overline{c})) = 0$ for all i and $b - \phi_i^*(\overline{c}) = 0$ for all i. Therefore, as b tends to $\overline{c}$, the system tends towards $\frac{0}{0}$ and therefore the Lipshcitz condition does not hold in a neighborhood of $\overline{c}$. Also, many commonly used methods to solve differential equations with boundary conditions require the system to be Lipschitz. In the asymmetric auction model, however, because the

Lipschitz condition fails alternative methods have to be developed to prove that the equilibrium is unique and to compute the equilibrium.

The proof of the uniqueness of equilibrium is based on Bajari (1997). LeBrun (1995b) demonstrated that there is a unique equilibrium under the assumption that $\{F_1, F_2, ..., F_N\} = \{G_1, G_2\}$, that is, when there are two distinct types of bidders in the auction. Maskin and Riley (1996a) have also proved that the equilibrium is unique in the case that there are N distinct bidders. This result was also established in Le Brun (1999).

The problem of computing the equilibrium to the asymmetric auction model has also been studied by Marshall, Muerer, Richard and Stromquist (1994) and Riley and Li (1997). Armantier, Florens and Richard (1997) have also proposed methods for computing the equilibrium to auction models. The algorithms described in this paper have been used for structural estimation of asymmetric auction models by Bajari (1997) and Bajari and Ye (2000).

3.1 Uniqueness of equilibrium

In this section, I will outline a short proof that demonstrates that the equilibrium is unique. In what follows, the key to the proof will be to show that there is only one value of β that satisfies (ii) of Theorem 2. In this section, a summary of the main arguments is outlined. A complete proof is presented in Appendix A.

Suppose, by way of contradiction, that there are two solutions to the system of ordinary differential equations which correspond to two different equilibria. Let β_1 and β_2 be the two different boundary conditions that satisfy part (ii) of Theorem 2 and suppose without loss of generality that $\beta_1 < \beta_2$.[4] Let $\phi_i(b;\beta_1)$ denote the inverse bidding strategy associated with the boundary condition β_1 and let $\phi_i(b;\beta_2)$ be the boundary condition associated with the inverse bidding strategy β_2. There are three steps in the proof.

First Step Demonstrate that for all i and all $b < \overline{c}$, $\phi_i(b;\beta_1) > \phi_i(b;\beta_2)$.

If the proposition in the first step was not true, there would exist a minimal value of b, call it b^*, such that $\phi_i(b^*;\beta_1) = \phi_i(b^*;\beta_2)$. Since $\phi_i(b;\beta_2)$ crosses $\phi_i(b;\beta_1)$ from below, it must be the case that $\phi_i'(b^*;\beta_1) < \phi_i'(b^*;\beta_2)$. Using this inequality and equation (6) it is simple to show that for some j, $\phi_i(b^*;\beta_1) < \phi_i(b^*;\beta_2)$ which contradicts that b^* is the first place where the solutions cross.

Second Step Demonstrate that for all i, $\lim_{b\uparrow\overline{c}} = \frac{N}{N-1}$.

This fact follows in a straightforward way by applying L'Hopital's rule to equation (6).

Third Step Show that for some b, $\phi_i(b;\beta_1) < \phi_i(b;\beta_2)$, which is a contradiction of the result in the first step.

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[4] Since the singularity only occurs at $\overline{c}$, arguments based on standard Lipschitz conditions guarantee if $\beta_1 = \beta_2$ then the equilibria would coincide for all $b < \overline{c}$. This would be a contradicton.

The probability that player i wins the auction with a bid of b is $Q(b;\beta_1) = \prod_{j\neq i}(1 - F_j(\phi_j(b;\beta_1))$ in the equilibrium with boundary condition β_1 and $Q(b;\beta_2) = \prod_{j\neq i}(1 - F_j(\phi_j(b;\beta_2))$ in the equilibrium with boundary condition β_2. By the result in the first step, it follows immediately that $Q(b;\beta_1) < Q(b;\beta_2)$. By a straightforward application of L'Hopital's rule and the fact from the second step that $\lim_{b\uparrow\overline{c}} = \frac{N}{N-1}$ one can show $\lim_{b\uparrow\overline{c}} \frac{Q(b;\beta_1)}{Q(b;\beta_2)} = 1$. Since $\frac{Q(b;\beta_1)}{Q(b;\beta_2)} < 1$ for all $b < \overline{c}$ (which follows from the first step) there exists a b such that $\frac{d}{db}\frac{Q(b;\beta_1)}{Q(b;\beta_2)} > 0$. By manipulating equation (6) it follows that $\phi_i(b;\beta_1) < \phi_i(b;\beta_2)$ which is the desired contradiction.

Theorem 3 *Suppose that assumptions 1 and 2 hold. Then there is a unique equilibrium.*

3.2 Computation of the equilibrium

In this section, I discuss three algorithms that can be used to compute the equilibrium inverse bid functions. The first algorithm was originally proposed by Maskin and Riley (1993) and was generalized by Riley and Li (1997). The idea behind this algorithm is to find the value of β that satisfies (ii) of Theorem 2. Since equation (6) is Lipschitz at the boundary condition (ii), if β is known standard methods in numerical ordinary differential equations can be used to compute numerical solutions to equation (6). A disadvantage of this algorithm is that it is rather slow to converge. In applied work, such as in Bajari (1997) and Bajari and Ye (2000), it may be necessary to evaluate the inverse bid function several hundred thousand times. A second algorithm used in Bajari (1997) starts with an initial guess that the bid function is for firms to bid their cost. Bajari (1997) then has firms make a sequence of best responses starting from this initial guess. In many cases, this algorithm converges rapidly, but convergence is not guaranteed. The third algorithm which is used in Bajari and Ye (2000) makes a polynomial approximation to the inverse bid functions. This algorithm is very fast if one has good starting values.

3.3 First algorithm

The idea behind the first algorithm is to find the value of β that satisfies (ii) of Theorem 2. For a given value of $\widetilde{b}$, $\underline{c} < \widetilde{b} < \overline{c}$, define the functions $\widetilde{\phi}_i(b;\widetilde{b})$ as solutions to the initial value problem:

$$\begin{aligned} \widetilde{\phi}_i'(b;\widetilde{b}) &= \frac{1 - F_i(\widetilde{\phi}_i(b;\widetilde{b}))}{(N-1)f_i(\widetilde{\phi}_i(b;\widetilde{b}))}\left[\frac{-(N-2)}{b - \widetilde{\phi}_i(b;\widetilde{b})} + \sum_{j\neq i}\frac{1}{b - \widetilde{\phi}_j(b;\widetilde{b})}\right] \qquad (7) \\ \widetilde{\phi}_i(\widetilde{b};\widetilde{b}) &= \underline{c} \quad \text{for } i = 1, ..., N \end{aligned}$$

The equilibrium inverse bid functions are solutions to the initial value problem defined by (7) when $\widetilde{b} = \beta$.

In the general N bidder case, define the set of functions S as follows:

$$S \equiv \{s : \text{s is } C^1,\ s : [\underline{c}, \overline{c}] \rightarrow [\underline{c}, \overline{c}] \text{ and } s(b) < b \text{ for all } b < \overline{c}\}.$$

In Appendix B, I establish that if $\widetilde{b} \in [\beta, \overline{c}]$, then the functions $\widetilde{\phi_i}(b; \widetilde{b})$ are in S. On the other hand, if $\widetilde{b} < \beta$, Appendix B demonstrates that the $\widetilde{\phi_i}(b; \widetilde{b})$ are not in S and therefore the solution diverges. As Appendix B demonstrates, standard methods in the numerical analysis of ordinary differential equations can be used to determine for a given $\widetilde{b}$ if the $\widetilde{\phi_i}(\widetilde{b}; \widetilde{b})$ are in the set S or not by assessing whether or not the solution to (7) diverges. (This property has been observed previously in Bajari (1997) and Riley and Li (1997)).

For a given tolerance ϵ, it is possible to find an interval (b_{low}, b_{high}) of length less than ϵ that contains β by using the following algorithm:

1. Fix $b_{low} = \underline{c}$ and $b_{high} = \overline{c}$.
2. Set $b_{guess} = \frac{1}{2}(b_{low} + b_{high})$.
3. Determine whether the system $\widetilde{\phi_i}(b; b_{guess})$ diverges, that is whether it is in S.
4. If $\widetilde{\phi_i}(b; b_{guess})$ is in S then set $b_{high} = b_{guess}$.
5. If $\widetilde{\phi_i}(b; b_{guess})$ is not in S then set $b_{low} = b_{guess}$.
6. If $b_{high} - b_{low} < \epsilon$ stop. Otherwise, go to step 2.

After the interval (b_{low}, b_{high}) has been found, set $\widetilde{b} = \frac{1}{2}(b_{low} + b_{high})$. If the distance between β in theorem 3 and $\widetilde{b}$ is small, the solution to the initial value problem 3.1 and 3.2 $\widetilde{\phi_i}(b; \widetilde{b})$ will be close to the equilibrium solution to the initial value problem $\widetilde{\phi_i}(b; \beta)$.

3.4 Second algorithm

A second method for computing the equilibrium to the asymmetric auction involves making an initial guess that in equilibrium, firms bid their cost. That is, we guess firms follow the strategy $\phi_i^0(b) = b$. Let $Q_i(b; \{\phi_j^0(b)\}_{j \neq i})$ be the probability that firm i is the low bidder given that all firms $j \neq i$ follow this strategy then:

$$\begin{aligned} Q_i(b; \{\phi_j^0(b)\}_{j \neq i}) &= \prod_{j \neq i} (1 - F_j(\phi_j^0(b))) \qquad (8) \\ &= \prod_{j \neq i} (1 - F_j(b)) \end{aligned}$$

Let $\phi_i^1(b)$ be a best response to a profile of strategies $\{\phi_j^0(b)\}_{j \neq i}$ by firm i. Then $\phi_i^1(b)$ will satisfy:

$$(b-\phi_i^1(b))Q_i(b;\{\phi_j^0(b)\}_{j\neq i})+Q_i'(b;\{\phi_j^0(b)\}_{j\neq i})=0 \qquad \text{for } i=1,\ldots,N. \quad (9)$$

$$\phi_i^1(b)=b+\frac{Q_i'(b;\{\phi_j^0(b)\}_{j\neq i})}{Q_i(b;\{\phi_j^0(b)\}_{j\neq i})} \qquad \text{for } i=1,\ldots,N.$$

Similarly, recursively define $\phi_i^{n+1}(b)$ by:

$$\phi_i^{n+1}(b)=b+\frac{Q_i'(b;\{\phi_j^n(b)\}_{j\neq i})}{Q_i(b;\{\phi_j^n(b)\}_{j\neq i})} \quad \text{for } i=1,\ldots,N. \qquad (10)$$

If F and its higher order derivatives can be evaluated with high accuracy, I have found it is possible to use the formula in equation (10) to compute the inverse bid function with high accuracy in a neighborhood of $\overline{c}$. While there is no theoretical guarantee that the solutions in (10) will converge, I have found in practice with truncated normal distributions, convergence is very fast in a neighborhood of $\overline{c}$.[5],[6],[7]

3.5 Third algorithm

The third algorithm approximates the inverse bid functions using a flexible functional form such as a high order polynomial and then finds a set of coefficients that approximately satisfy the differential equation (6) and the boundary conditions (i) and (ii) of Theorem 2. My third approach is closely related to methods proposed by Armantier, Florens and Richard (1997). Let $\underline{b}$ be the lowest bid that is submitted in equilibrium. I shall assume that the bid function for player i takes the form:

$$\phi_i(b;\underline{b},\alpha)=\underline{b}+\widetilde{\phi}_i(b;\alpha) \qquad (11)$$

where

$$\widetilde{\phi}_i(b;\alpha)=\sum_{k=0}^{5}\alpha_{i,k}(b-\underline{b})^k \qquad (12)$$

In equilibrium, firm i's first order condition for profit maximization can be written as:

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[5] Athey (1997) has reported that these solutions are numerically unstable in computational experiments. Unlike Athey (1997), however, in the computational experiments conducted here the strategy space and type space were not discretized. Since the F_i are truncated normal, the higher order derivatives that can be expressed in closed form, and therefore the sequence of best responses can also be express in closed form.

[6] Bajari (1997) also solves for the inverse bid functions by using differential-algebraic equations solvers. Publicly available packages such as DDASSL have been used with some success in this problem. Due to the technical nature of the theory behind these solvers, they are not discussed in this paper. The interested reader is referred to Petzold (1983) for a discussion.

[7] Another method that can be used to compute bid functions is to discretize the set of types and/or the set of bids. This is the approach followed in Athey (1997). She is able to achieve convergence by using a simple iterative method. If both the space of type and bids are discretized, it is possible to adapt the methods described in McKelvy and McLennan (1996) to compute the equilibrium.

$$1 + \sum_{j \neq i} \frac{f_j(\phi_j(b))\phi_j'(b)(b - \phi_i(b))}{(1 - F_j(\phi_j(b)))} = 0 \tag{13}$$

Define the function $G_i(b, \alpha)$ as:

$$G_i(b; \underline{b}, \alpha) = 1 + \sum_{j \neq i} \frac{f_j(\phi_j(b; \underline{b}, \alpha))\phi_j'(b; \underline{b}, \alpha)(b - \phi_i(b; \underline{b}, \alpha))}{(1 - F_j(\phi_j(b, \alpha)))} \tag{14}$$

I evaluate the first order conditions on a grid of points that is uniformly spaced between $\underline{b}$ and $\overline{c}$. Let $ngrid$ denote the number of points in the grid, define the k^{th} point in the grid $bgrid_k$ as:

$$bgrid_k = \underline{b} + \frac{k * (\overline{b} - \overline{c})}{ngrid + 1} \tag{15}$$

Define $H(\underline{b}, \alpha)$ as:

$$H(\underline{b}, \alpha) = \sum_i \sum_k G_i(bgrid_k, \underline{b}, \alpha)^2 + \sum_i (\underline{c} - \phi_i(\underline{b}; \underline{b}, \alpha))^2 + \sum_i (\overline{c} - \phi_i(\overline{c}; \underline{b}, \alpha))^2 \tag{16}$$

If H is equal to zero then equation (6) is satisfied at the grid points and the boundary conditions of Theorem 2 are also satisfied. A standard non-linear least squares algorithm can be used to find the value of α and $\underline{b}$ that minimize H.

An advantage of the third algorithm is that with good starting values it is very fast. In structural estimation, the econometrician will often evaluate changes in the likelihood function given a small change in the structural parameters. Therefore, good starting values will be available at many parameter values and using the third algorithm can dramatically reduce the computational time for structural estimation as compared to the first algorithm. In practice, I have found that even third or fourth order polynomials can give solutions that are accurate to 5 or more significant digits.

4 Properties of bidding strategies

This section uses the computational algorithms described in Section 3 to study the equilibrium bidding strategies. The bid functions are graphed, comparative statics properties of the bid functions are summarized and the difference between competitive and collusive bidding is explored.

Let the distribution of costs be truncated normal with support $[\underline{c}, \overline{c}] = [2, 8]$. The probability density function for bidder i, $f_i(c_i; \theta_i)$ has firm i specific parameters $\theta_i = (\mu_i, \sigma_i)$ and satisfies:

$$f_i(c_i; \theta_i) \propto \begin{cases} \frac{1}{(2\pi\sigma_i^2)^{\frac{1}{2}}} \exp\left(-\frac{(c_i - \mu_i)^2}{2\sigma_i^2}\right) \text{ if } \underline{c} < c_i < \overline{c} \\ 0 \text{ otherwise} \end{cases} \tag{17}$$

Suppose there are three bidders and that the firm specific parameters are $\mu_1 = 4.0$, $\mu_2 = 5.0$, $\mu_3 = 6.0$ and $\sigma_1 = \sigma_2 = \sigma_3 = 1.5$. The bid functions for all three firms are graphed in Figure 1. For a given cost draw, firm 1 bids the highest, firm 2 the second highest, and firm 3 the lowest. The markup of firm 1 over cost is the largest and for all firms, the markup tends to decrease as costs tend towards $\overline{c}$. Despite the seemingly complicated form of the system of differential equations in Theorem 2, the bid functions are remarkably simple.

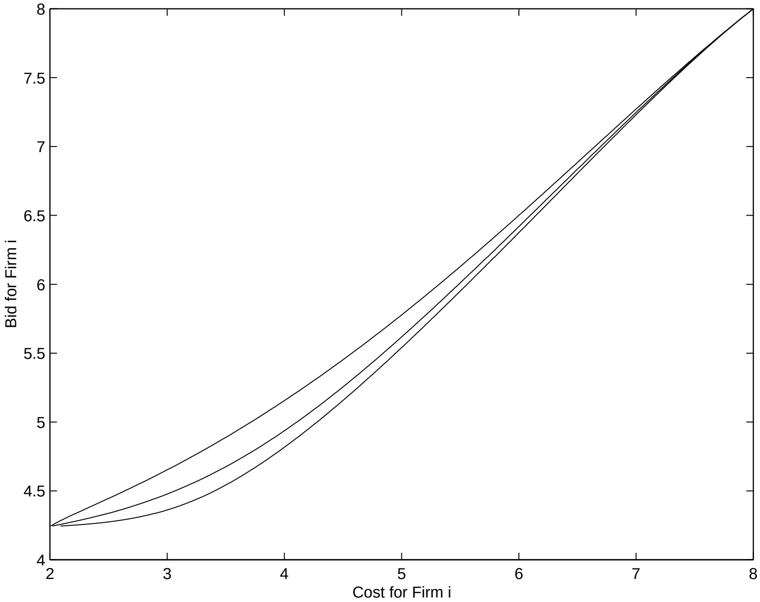

Figure 1. Bid functions

The first computational experiment I perform is to demonstrate how the bid functions, probability of winning and expected profit depend on the parameters $\theta = (\theta_1, ..., \theta_N)$. I compute the inverse equilibrium bid functions allowing the μ_i vary between 4 and 6. I assume that $\sigma_i = \sigma$ for all i and that the parameter σ varies between 1 and 2.

A first order polynomial approximation to the bid functions is summarized in Table 1 when there are 3 firms and in Table 2 when there are 4 firms. For instance, the first column of Table 1 corresponds to the first order approximation to firm 1's bid function in equation (18):

$$b_1 = 0.1312 * \mu_1 + 0.2079 * \mu_2 + 0.2079 * \mu_3 + \sigma * -0.0021 + 0.5522 * c_1 \quad (18)$$

The second column and third columns display the coefficients for profit and the probability of winning.

In the games with three firms and four firms, the bid, expected profit and probability of winning are all increasing in μ_1, μ_2, μ_3 and (if there are 4 bidders) μ_4. The monotonicity of profits in μ_1 results from when firm 1 is expected

Table 1. Bidding with 3 firms

Parameter	Bid	Expected profit	Probability of winning
μ_1	0.1312	0.1111	0.0853
μ_2	0.2079	0.1440	0.0857
μ_3	0.2079	0.1440	0.0857
σ	-0.0021	-0.0045	-0.0007
c_1	0.5522	-0.3028	-0.1871

Table 2. Bidding with 4 firms

Parameter	Bid	Expected profit	Probability of winning
μ_1	0.0424	0.0602	0.0595
μ_2	0.0992	0.0815	0.0601
μ_3	0.0992	0.0815	0.0601
μ_4	0.0992	0.0815	0.0601
σ	0.1083	-0.0599	-0.0348
c_1	0.7434	-0.2217	-0.1721

to have a higher cost draw, competition softens. As the number of bidders increases from 3 to 4, the coefficient on c_1 increases from 0.5522 to 0.7434 which indicates that firm 1 bids more aggressively as the number of bidders increases. This result is reminiscent of many models of imperfect competition, such as the Cournot model, where markups decrease as the number of players rise.

Next, the comparative statics of collusive bidding are studied in a computational experiment. There are many possible models of collusion in auctions. Bidders could rotate winning according to the “phases of the moon”, establish territories or make side payments. I study a simple model of collusion where firm 1 and 2 act as an “efficient” cartel. Firm 1 and 2 each make a cost draw c_1 and c_2 before the auction begins. The cartel is assumed to operate efficiently so that the low cost firm will submit a “real” bid for the cartel and the other firm will either refrain from bidding or submit a higher “phony” bid. To study the equilibrium with a cartel, I only need to make a simple modification to the asymmetric auction model. The cartel is modeled as a single agent who maximizes expected profit with cost $c_c = \min\{c_1, c_2\}$. I compute the equilibrium to the model on the same grid of parameters as in Tables 1 and 2. A first order approximation to the bid functions is displayed in Table 3.

In Table 3, the bid for both the cartel and firm 3 are both monotone increasing in all the underlying parameters. Cartel bidding differs from competitive bidding in at least two ways. First, the slope of the cartel’s bid function in μ_3 is 0.2925, which is considerably higher than the slope under the hypothesis of competition, 0.2079. A second difference between competitive and collusive bidding is that the distribution of bids in a competitive model will satisfy a property I will call exchangeability whereas the model of collusive bidding will not. Notice that in Tables 1 and 2, the coefficient on μ_2, μ_3 and (if there are four bidders) μ_4 are

Table 3. Bidding with a cartel

Parameter	Bid for cartel	Bid for firm 3
μ_1	0.1093	0.1709
μ_2	0.1093	0.1709
μ_3	0.2925	0.1595
σ	0.3074	0.2210
c	0.6041	0.6064

all equal. This implies, in the case where there are three bidders, firm 1's bid function is unchanged if $(\mu_2, \mu_3) = (5, 6)$ or $(\mu_2, \mu_3) = (6, 5)$. However, from Table 3, in the case of collusion, firm 1's observed bids clearly would not satisfy this property. When firm 1 is the low cost firm (and therefore submits a "real" bid for the cartel), firm 1's bid function would clearly differ depending upon whether $(\mu_2, \mu_3) = (5, 6)$ or $(\mu_2, \mu_3) = (6, 5)$.

Next, I formally define exchangeability. First, consider a simple thought experiment where we change the cost parameters of firms 1 and 2. Initially, set $\theta_1 = \theta_A$ and $\theta_2 = \theta_B$ and in the second case switch firm 1 and firm 2's cost parameters so that $\theta_1 = \theta_B$ and $\theta_2 = \theta_A$. When there is no cartel, the bid functions clearly satisfy:

$$b_1(c; \theta_A, \theta_B, \theta_3, ..., \theta_N) = b_2(c; \theta_B, \theta_A, \theta_3, ..., \theta_N) \tag{19}$$

That is, if I exchange the cost parameters of firm 1 and firm 2 holding all else fixed then the bid functions similarly permute.

More generally, I introduce the notion of a permutation. Let ρ be a one to one mapping from the set $\{1, ..., N\}$ onto itself. We will refer to such a map as a permutation since it merely rearranges the indices. If the equilibrium distribution of bids is exchangeable that means for any permutation ρ and any index i

$$b_i(c; \theta_1, \theta_2, \theta_3, ..., \theta_N) = b_{\rho(i)}(c; \theta_{\rho(1)}, \theta_{\rho(2)}, \theta_{\rho(3)}, ..., \theta_{\rho(N)}) \text{ for all } i \tag{20}$$

That is to say, if the parameters for all of the bidders are permuted by ρ, then the distribution of bids can also be permuted by ρ.

Theorem 4 *With no cartel, the equilibrium distribution of bids is exchangeable. That is, for all permutations ρ and any index i, $b_i(c; \theta_1, \theta_2, \theta_3, ..., \theta_N) = b_{\rho(i)}(c; \theta_{\rho(1)}, \theta_{\rho(2)}, \theta_{\rho(3)}, ..., \theta_{\rho(N)})$.*

Proof. This follows directly from the characterization of equilibrium in Theorem 2 and the fact that the equilibrium is unique. Q.E.D.

In our example where a cartel maximizes joint profits, the bidding behavior of the cartel members will fail to be exchangeable. Therefore, exchangeability is a necessary condition that holds in our model of competition and will not hold in all models of collusion. If we see a failure of exchangeability in the empirical distribution of bids, we can conclude that a necessary condition of competitive bidding has failed. In applied work that documents the bidding behavior of cartels, such as Porter and Zona (1993,1999), the author's descriptions of bidding behavior can be interpreted as a failure of exchangeability. In Porter and Zona (1993), the authors note that the residuals to the bid functions of the cartel members are much more highly correlated than non-cartel members and that participation in the auction is far more correlated between cartel members than non-cartel members. These patterns in bidding are clearly a failure of exchangeability. Bajari and Ye (2000) also use exchangeability to test for collusive bidding.

Another use of our computational tools is that we can assess the damages from bid rigging. Let's suppose that all firms have a $\mu_i = 5$ and that $\sigma = 1.5$. Also, suppose that firm 1 is the low cost firm who wins the auction. A simple computation from Tables 1 and 3 shows that the difference between the competitive and collusive bid (up to a first order approximation) is:

$$\text{collusive bid-competitive bid} = 0.2847 + 0.0519 * c$$

For instance, when the cost of firm 1 is 5, the markup is 0.5442 cents higher, or an increase of more than 10 percent.

The methods in this paper could be applied to the problem of determining damages due to bid rigging. If it is possible to recover, at least approximately, the value of θ either from court testimony or econometric analysis, the methodology presented in this paper would allow us to simulate a competitive equilibrium in the market as a basis for determining damages. If competitive bidding had not occurred in the market for some time, simulation of the asymmetric auction model might be the only available approach to assessing damages.

5 Conclusion

In this research, I studied a model of competitive bidding for procurement contracts. I have argued that realistic models of procurement contracting must allow for asymmetric bidders. Asymmetries can occur because firms have different locations, differences in managerial efficiency or collusion is present. Studying models with asymmetric bidders in the first price sealed bid auction is technically difficult. The inverse bid functions can be characterized by a system of ordinary differential equations with boundary conditions. However, since the system is not Lipschitz, standard approaches to computing the equilibrium and to proving uniqueness do not apply. This research proved the uniqueness of equilibrium and described three approaches to computing the equilibrium inverse bid functions. As an application, I numerically studied the comparative statics of bidding and compared competitive and collusive bidding. I find that firms bid more aggressively as the number of bidders increase and as their competitors become more efficient. Also, I showed that an important difference between competitive and collusive bidding is that competitive bid functions always satisfy a property I call exchangeability while collusive models do not always satisfy this property. The computational algorithms developed in this paper can be used in structural estimation or to assess the damages from bid-rigging.

Appendix A. Uniqueness of equilibrium

The uniqueness of the equilibrium is proved in three steps as described in Section 3. The first step is to show that for all i, the inverse bid functions satisfy $\lim_{b\uparrow\overline{c}} \phi_i'(b) = \frac{N}{N-1}$. To make the notation more compact, I shall often write $\phi_i'(\overline{c})$ in lieu of $\lim_{b\uparrow\overline{c}} \phi_i'(b) = \frac{N}{N-1}$.

Lemma 5 *Suppose that Assumption 1 and 2 hold. Let ϕ_i for $i = 1, .., N$ be a set of inverse equilibrium bid functions. Then* $\lim_{b\uparrow\overline{c}} \phi_i'(b) = \frac{N}{N-1}$.

Proof. The proof follows from applying L'Hopital's rule to equation (6). Since $1 - F_j(\phi_j(b))$ tends to zero for all j as $b \uparrow \overline{c}$ and since $b - \phi_i(b)$ also tends toward zero as $b \uparrow \overline{c}$ the terms in equation (6) are of the form $\frac{0}{0}$ and L'Hopital's rule must be used to compute the derivative. Applying L'Hopital's rule to the system yields

$$\phi_i'(\overline{c}) = \frac{(N-2)\phi_i'(\overline{c})}{(N-1)(1-\phi_i'(\overline{c}))} - \sum_{j\neq i} \frac{\phi_i'(\overline{c})}{(1-\phi_j'(\overline{c}))(N-1)} \text{ for all } \ i = 1, ..., N.$$

The result then follows by straightforward algebra. Q.E.D.

Now, let's assume by way of contradiction that the equilibrium in not unique. Let β_1 and β_2 be the two different boundary conditions associated with the distinct equilibria that satisfy part (ii) of Theorem 2 and suppose that $\beta_1 < \beta_2$. Let $\phi_i(b; \beta_1)$ denote the inverse bidding strategy associated with the boundary condition β_1 and let $\phi_i(b; \beta_2)$ be the boundary condition associated with the inverse bidding strategy β_2. The next step in the proof is to demonstrate that for all i and all b in the domain of both functions, $\phi_i(b; \beta_1) > \phi_i(b; \beta_2)$.

Lemma 6 *For all i, if $\beta_1 < \beta_2$ then for all $b, \beta_2 < b < \overline{c}$, $\phi_i(b; \beta_1) > \phi_i(b; \beta_2)$.*

Proof. Suppose by way of contradiction that this is not true. Then there exists an i and a smallest value of b such that $\phi_i(b; \beta_2)$ crosses $\phi_i(b; \beta_1)$ from below. Suppose without loss of generality that $i = 1$ and the point where the crossing first occurs is at b^*. Since $\phi_1(b; \beta_2)$ crosses $\phi_1(b; \beta_1)$ from below at b^* it must be the case that $\phi_1'(b; \beta_2) \geq \phi_1'(b; \beta_1)$. By equation (6) the following set of inequalities follows immediately:

$$\frac{1 - F_1(\phi_1(b; \beta_2))}{(N-1)f_1(\phi_1(b; \beta_2))}\{\frac{-(N-2)}{b - \phi_1(b; \beta_2)} + \sum_{j\neq 1} \frac{1}{b - \phi_J(b; \beta_2)}\}$$

$$\geq \quad \frac{1 - F_1(\phi_1(b; \beta_1))}{(N-1)f_1(\phi_1(b; \beta_1))}\{\frac{-(N-2)}{b - \phi_1(b; \beta_1)} + \sum_{j\neq 1} \frac{1}{b - \phi_j(b; \beta_1)}\},$$

and since $\phi_1(b^*; \beta_1) = \phi_1(b^*; \beta_2)$,

$$\frac{-(N-2)}{b^* - \phi_1(b^*; \beta_2)} + \sum_{j\neq 1} \frac{1}{b^* - \phi_j(b^*; \beta_2)} \geq \frac{-(N-2)}{b^* - \phi_1(b^*; \beta_1)} + \sum_{j\neq 1} \frac{1}{b^* - \phi_j(b^*; \beta_1)},$$

once again, since $\phi_1(b^*;\beta_1) = \phi_1(b^*;\beta_2)$,

$$\sum_{j\neq 1} \frac{1}{b^* - \phi_j(b^*;\beta_2)} \geq \sum_{j\neq 1} \frac{1}{b^* - \phi_j(b^*;\beta_1)} \, .$$

Therefore, for some $j \neq 1$, $\frac{1}{b^*-\phi_j(b^*;\beta_2)} \geq \frac{1}{b^*-\phi_j(b^*;\beta_1)}$ and hence straightforward algebra implies that $\phi_j(b^*;\beta_1) \leq \phi_j(b^*;\beta_2)$. If $\phi_j(b^*;\beta_1) = \phi_j(b^*;\beta_2)$ for all j, then since the system is Lipschitz at b^*, $\phi_j(b^*;\beta_1)$ and $\phi_j(b^*;\beta_2)$ are equal in a neighborhood of b^* which implies that b^* is not the first point at which the solutions crossed. If $\phi_j(b^*;\beta_1) < \phi_j(b^*;\beta_2)$ the original contradiction hypothesis is violated. Q.E.D.

Using lemma 5 and 6, it is now possible to complete the proof of Theorem 3.

Proof of Theorem 3. Suppose once again that there are two distinct equilibria parameterized by the boundary conditions β_1 and β_2, $\beta_1 < \beta_2$. The following set of inequalities is then straightforward and holds for all i:

$$\begin{aligned} \phi_i(b;\beta_1) \quad &> \quad \phi_i(b;\beta_2) \\ &\Longrightarrow \quad 1 - F(\phi_i(b;\beta_2) > 1 - F(\phi_i(b;\beta_1) \\ &\Longrightarrow \quad \frac{1 - F(\phi_i(b;\beta_1)}{1 - F(\phi_i(b;\beta_2)} < 1. \end{aligned}$$

Therefore, by the definition of $Q_i(b;\beta_1)$ and $Q_i(b;\beta_2)$ it follows immediately that for all i the ratio $\frac{Q_i(b;\beta_1)}{Q_i(b;\beta_2)} < 1$. By lemma 1 and L'Hopital's rule it follows immediately that $\lim_{b\uparrow\bar{c}} \frac{Q_i(b;\beta_1)}{Q_i(b;\beta_2)} = 1$. Obviously then, for some $b < \bar{c}$, $\frac{d}{db}\frac{Q_i(b;\beta_1)}{Q_i(b;\beta_2)} > 0$ since the ratio is always less than 1 but converges to 1 in the limit. Computing the derivative implies that for some b:

$$\frac{Q_i'(b;\beta_1)Q_i(b;\beta_2) - Q_i'(b;\beta_2)Q_i(b;\beta_1)}{(Q_i(b;\beta_2))^2} > 0$$

$$\begin{aligned} -\frac{Q_i'(b;\beta_2)}{Q_i(b;\beta_2)} \quad &> \quad -\frac{Q_i'(b;\beta_1)}{Q_i(b;\beta_1)} \text{by multiplying the inequality by } \frac{(Q_i(b;\beta_2))^2}{Q_i(b;\beta_2)Q_i(b;\beta_1)} \\ -\frac{Q_i(b;\beta_1)}{Q_i'(b;\beta_1)} \quad &> \quad -\frac{Q_i(b;\beta_2)}{Q_i'(b;\beta 2)} \text{since both sides of the above ratio are positive.} \\ b - \phi_i(b;\beta_1) \quad &> \quad b - \phi_i(b;\beta_2) \text{by the first order conditions} \\ \phi_i(b;\beta_1) \quad &< \quad \phi_i(b;\beta_2) \end{aligned}$$

This violates Lemma 2 and a contradiction is therefore obtained. Q.E.D.

Appendix B. Computation of equilibrium

Recall that the set of functions S is defined as:

$S \equiv \{s : \text{s is } C^1,\ s : [\underline{c},\bar{c}] \to [\underline{c},\bar{c}] \text{ and } s(b) < b \text{ for all } b < \bar{c}\}$.

Let β be the scalar that satisfies the boundary condition (ii) of Theorem 2 for the equilibrium inverse bid functions. This appendix will first establish that if $\widetilde{b} \in [\beta, \overline{c}]$, then the function $\widetilde{\phi}_i(b;\widetilde{b})$, as defined by the initial value problem (7), is in S and if $\widetilde{b} < \beta$, $\widetilde{\phi}_i(b;\widetilde{b})$ is not in S and therefore the solution diverges.

Lemma 7 *If* $\widetilde{b} \in [\beta, \overline{c}]$, *then the function* $\widetilde{\phi}_i(b;\widetilde{b})$ *is in* S .

Proof. Suppose, by way of contradiction that $\widetilde{\phi}_i(b;\widetilde{b})$ is not in S. Then $\widetilde{\phi}_i(b;\widetilde{b})$ must cross $\widetilde{\phi}_i(b;\beta)$ from below. Assume that the crossing occurs at first at b^* for player i's inverse bid functions. Since $\widetilde{\phi}_i(b;\widetilde{b})$ crosses $\widetilde{\phi}_i(b;\beta)$ from below it must be the case that $\widetilde{\phi}_i'(b^*;\widetilde{b}) \geq \widetilde{\phi}_i'(b^*;\beta)$. The rest of the argument follows as in Lemma 2. Q.E.D.

Lemma 8 *If* $\widetilde{b} \in [\underline{c}, \beta)$, *then* $\widetilde{\phi}_i(b;\widetilde{b})$ *is not in* S.

Proof. The proof considers the following two cases: $\lim_{b\uparrow\overline{c}} \widetilde{\phi}_i(b;\widetilde{b}) < \overline{c}$ for some i and $\lim_{b\uparrow\overline{c}} \widetilde{\phi}_i(b;\widetilde{b}) = \overline{c}$ for all i.

Case #1: $\lim_{b\uparrow\overline{c}} \widetilde{\phi}_i(b;\widetilde{b}) < \overline{c}$ for some i.

Since at the equilibrium solution, $\lim_{b\uparrow\overline{c}} \widetilde{\phi}_i(b;\beta) = \overline{c}$, it must be the case that $\widetilde{\phi}_i(b;\beta)$ crosses $\widetilde{\phi}_i(b;\widetilde{b})$ from below. Let b be the first place where this crossing takes place. It must be the case that $\widetilde{\phi}_1'(b;\beta) \geq \widetilde{\phi}_1'(b;\widetilde{b})$. This will lead to a contradiction as in Lemma 2.

Case #2: $\lim_{b\uparrow\overline{c}} \widetilde{\phi}_i(b;\widetilde{b}) = \overline{c}$ for all i.

Arguing as in Lemma 1, $\lim_{b\uparrow\overline{c}} \phi_i'(b;\widetilde{b}) = \frac{N}{N-1}$. Furthermore, it must be the case that for $b < \overline{c}$, $\widetilde{\phi}_i(b;\widetilde{b}) > \widetilde{\phi}_i(b;\beta)$. The proof then proceeds along the lines of the proof of Theorem 3, with $\widetilde{b}$ taking the role of β_1 and β taking the role of β_2. Q.E.D.

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