ECONOMETRICS OF SEALED-BID AUCTIONS.

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1. Introduction.

Recent empirical and anecdotal evidence suggests that different auction rules generate important differences in bidding behavior. Milgrom (1996) compares auctions for the sale of the radio spectrum held in the United States, New Zealand and Australia. The Australian and New Zealand auctions were characterized by wildly different prices for similar bandwidths. In the United States price differences between identical bandwidths were of a far smaller magnitude. Milgrom argues that the different rules of the auctions can account for the different bidding behavior in the United States, Australia, and New Zealand. Kagel (1995) surveys a wealth of experimental evidence that documents how different auction rules change subjects' bidding behavior.

Applied work in auctions clearly needs to take account of the "rules of the game". Game theoretic models of auctions allow the economist to clearly specify the strategies and utility functions of the bidders. There now exists a large body of economic theory that studies the equilbria to a wide variety of auction games. (See Wilson (1992) and McAfee and McMillan (1987) for surveys.)

The standard auction models assume that the players do not know each other's valuations for the good being auctioned. The standard solution concept of Bayes-Nash equilibrium, if it exists, generates a distribution of observed bids resulting from every agent in the game maximizing utility. The equilibrium distribution of bids can be used as a likelihood function for applied work in economics. In particular, if the likelihood function can be evaluated efficiently, Markov chain Monte Carlo can be used to simulate the posterior distribution of model parameters. (See Geweke (1994) and Tierney (1994) for overviews of these methods.) The posterior distribution of model parameters allows the econometrician to use observed bids to learn about the economic primitives, such as the utility functions and the information structure that generated the bids.

Paarsch (1992) and Laffont, Ossard, and Vuong (1995) discuss estimation procedures applicable in first price sealed bid auctions, that is, auctions where all bidders submit sealed bids and the highest bidder wins the auction. The authors make a number of assumptions about the game to render their econometric problem tractable: all bidders are ex-ante identical, it is costless to prepare a bid, and the number of bidders is exogenously specified.

There are major technical hurdles for both theoretical and econometric analysis of first price auctions when these assumptions are dropped. Few results can establish existence, much less uniqueness of the equilibrium, outside of these restrictive assumptions. General existence and uniqueness results do not exist when the econometrician allows for endogenous participation of the bidders, models the information acquisition processes of the bidders or allows the bidders to participate in more than a single auction. See Maskin and Riley (1996 a,b), Athey (1997) and Bajari (1996,1997) for a discussion of existence and uniqueness results in first price auctions. Without an equilibrium distribution of bids, a likelihood function for applied work cannot be derived using game theory.

Even when the existence and uniqueness of equilibrium hold, computation of the equilibrium can be quite difficult. Several researchers have considered the problem of asymmetric auctions, that is, auction games where some bidders are known to have a different distribution of valuations before the auction occurs. Marshall et al (1995) and Riley and Lee (1995) have computed solutions to the asymmetric auction for two bidders and private values. Bajari (1996,1997) and Froeb et al (1997) have computed the solution for N bidders and private values. (Private values means that bidder i's utility does not depend on how bidder j values the item in the auction.) Efficient computation of the equilibrium is a necessary condition to evaluate the likelihood function efficiently. Efficient methods for computing the equilibria of more general first price auction games do not yet exist.

Structural estimation of auction models has several important applications. First, much testing of auction theory is done by testing reduced form implications of bidding behavior. In many cases, these tests have very low statistical power. Structural estimation often leads to more statistically powerful testing since the equilibrium is computed a part of the estimation procedure. By comparing the structural model to observed bidding behavior, new directions for research are often suggested. For instance, after estimating a symmetric bidding model, Laffont, Ossard, and Vuong (1995) find that their model fails to describe the bidding behavior of a large firm in the market. They conclude that asymmetries are an important direction for future research. Bajari (1997) finds, after estimating an independent private values model of bidding by construction firms, that bids are correlated between some firms in ways that are inconsistent with independent valuations. He concludes that modeling correlation in the information structure is a promising avenue for future research.

Second, structural models can be used to compare alternative auction formats. Donald and Paarsch (1997) use a structural model to derive the optimal reserve price for a timber auction. Bajari (1997) uses structural estimation to compare a first price auction to a second price auction.

Third, structural models can be used to infer collusion in procurement. Bid rigging is one of the most frequent violations of the Sherman act and is quite difficult to detect in practice. Bajari (1997) uses a structural model to compare the alternative hypotheses of competition and collusion by the two largest firms in the construction market he studies.

Finally, structural models can be used to assess the impact of mergers. Froeb et. al(1997) use structural models to simulate post merger bidding in forest timber sales.

This research discusses three technical challenges for structural estimation of auction models and proposes new methods to overcome these technical obstacles. A first difficulty is that many theoretical models are intractable analytically. For instance, Bajari (1997) in his study of road construction firms notes that capacity constraints can play an important role in firm decision making. However, such a model is probably hopelessly intractable to study by analytic methods.

A second difficulty is that the likelihood function for many auction models does not have full support. This will make the analysis very sensitive to outliers in the data.

A third problem is that the decision to bid in an auction is endogenous. Porter and Zoma (1997) study the bidding behavior of firms that supply milk to public schools in Ohio. They define the number of bid opportunities as the total number of auctions times the total number of firms in their data set. They note that only 2 percent of bid opportunities actually lead to submitted bids. Porter and Zoma estimate a selection equation to deal with this problem in their reduced form specification. A solution to this problem for structural modeling does not yet exist. These challenges are discussed in section 2 of this paper.

This research exploits recent advances in the computation of game theoretic equilibria to address the above problems. McKelvy and McLennan (1995) present a useful survey of the literature. Most structural estimation algorithms require that the econometrician solve for the equilibrium at least once for each parameter value at which the objective function or likelihood function is evaluated. Despite the obvious use of these algorithms, their importance has not been widely recognized. This research exploits an algorithm suggested by McKelvy and Palfrey (1995) to derive a likelihood function for a wide class of auction models. The posterior distribution of the parameters is then simulated using Markov chain Monte Carlo. Although the analysis of this paper is limited to auction econometrics, the methods have wider applications for structural estimation of games of incomplete information

McKelvy and Palfrey (1995,1997) adapt standard discrete choice models to study games in normal and extensive form. One model they study is called a logit or quantal response equilibrium. The player's problem is to choose one out of a finite set of possible strategies. The utility to the player from any strategy is the sum of the expected payoff in the game plus an extreme value disturbance drawn independently for each strategy. Best responses are probabilistic as opposed to deterministic as in most game theoretic analysis.

McKelvy and Palfrey's model does not abandon the assumption of equilibrium. Rather, it nests standard equilibrium models in a framework in which players estimate their equilibrium payoffs in an unbiased way. See McKelvy and Palfrey (1995,1997) for an extensive discussion of this concept. Their solution concept has a unique equilibrium and can often be computed efficiently.

In section 3, notation for a game theoretic model of auctions is presented. A Bayes-Nash equilibrium and quantal response equilibrium are defined and the likelihood function is then derived. An asymmetric auction is then studied. Both models generate likelihood functions that can be computed efficiently.

In section 4, equilibrium bidding behavior in a quantal response model is compared to the Bayes-Nash solution concept. The quantal response model generates a likelihood function with full support, unlike the Bayes-Nash model. A computational experiment is conducted to explore the sensitivity of these two models to the presence of an outlier in the data set. Parameter values are fixed and the bidding distributions are simulated for 100 auctions. A single outlier is then placed in the data set. The posterior distribution of parameters is simulated. Even with a large number of observed bids, the Bayes-Nash model is very sensitive to the inclusion of a single outlier. The quantal response model does not exhibit this sensitivity.

In section 5, an auction model with endogenous entry, uncertainty over the number of bidders, and asymmetric firms is studied. This model has not yet been studied in the theoretical literature since analytical characterization of the equilibrium appears to be quite difficult. The equilibrium is computed at approximately 20,000 parameter values to describe how the bidding behavior of the firms depends on the underlying parameters of the model.

A computational experiment is conducted to explore biases from not explicitly modeling entry. Parameter values are fixed and bids for 50 auctions are simulated. The posterior distribution for models both with and without entry is simulated. The simulation results show that parameter estimates from models that fail to model entry can be severely biased.

Section 2. Three Difficulties for Structural Auction Models.

In this section, I describe three technical difficulties faced in structural estimation of auction models. First, for many auction games, existence, uniqueness and characterization of equilibrium are open questions. Second, auction models often generate likelihood functions that do not have full support. The presence of a single outlier, even in a large data set, can bias parameter estimates. Third, in auction data sets, severe censoring often exists. Failure to model the entry decision can also bias parameter estimates.

Section 2.1. Difficulties with Computation and Characterization.

To fix ideas, consider a game with N contractors bidding on an indivisible public works project. The auction will be a first price auction, that is, the low bidder will win. Auctions are games with private information. Each contractor's cost c_i is a random variable with a cumulative distribution function $F_i(c_i|q)$ and probability distribution function $f_i(c_i|q)$ where q is a vector of parameters to be estimated. Before the auction takes place, each firm learns c_i, but is ignorant of c_j for j¹i. A strategy for firms in this game is a function $b_i(c_i|q)$. The bidding function specifies for all possible values of the random variable c_i how contractor i should bid.

If the bidding function $b_i(c_i|q)$ is strictly monotone and differentiable, it has an inverse $f_i(b_i|q)$ with derivative f_i¢$(b_i|q)$. The probability density function for b_i can then be written as $f_i(f_i(b_i|q))f_i$¢$(b_i|q)$. If the likelihood function can be evaluated efficiently, it is possible to use standard techniques in Markov chain Monte Carlo to simulate the posterior distribution of q conditional on the observed bids.

To evaluate the likelihood function, it is necessary to compute the inverse bidding strategy $f_i(b_i|q)$ and its derivative f_i¢$(b_i|q)$. Empirical work requires a flexible specification of the bidding game. Unfortunately, it is an open question whether existence and uniqueness of Bayes-Nash equilibrium holds in many auction games. For example, there are no general existence and uniqueness results for games with asymmetric bidders with correlated valuations, endogenous information acquisition or entry. (See Maskin and Riley (1997a,b), Athey (1997) and Bajari (1997) for discussions of the existence and uniqueness problems.)

Section 2.2 Zero Likelihoods.

Another challenge often faced in empirical work is that the likelihood for player i's bid, $f_i(f_i(b_i|q))f_i$¢$(b_i|q)$, will not have full support in many auction games. Many auction games give positive probability only to bids in some compact interval $[\underline{b}(q),\overline{b}(q)]\subset[0,\infty]$. (Since the lowest and highest bid in the support of all players can be shown to be equal in most auction models, we do not index $\underline{b}(q)$ and $\overline{b}(q)$ by a player specific subscript.) This condition rises from very robust economic reasoning. For instance, a firm with zero cost would never bid zero in equilibrium. Therefore, there is a lower bound $\underline{b}(q)$ on the value of bids that can be observed in equilibrium.

In general, the set of bids in the support, $[\underline{b}(q),\overline{b}(q)]$, depends on the value of the parameter q. In an asymmetric independent private values model, the value of $\underline{b}(q)$ decreases as the mean of $F_j(c_i|q)$ decreases or the standard deviation increases for any player j. If a bid b_i less than $\underline{b}(q)$ is present in the data set, due to a strategic mistake on the part of a bidder or measurement error in the data, it will not have positive probability. To give b_i positive probability, the mean of $F_j(c_j|q)$ must decrease or its standard deviation must increase for at least some j. Therefore, an outlier in the data set will cause the econometrician to underestimate the first moment and overestimate the second moment of $F_i(c_i|q)$ for one or more players i. As we will show below, even the inclusion of a single outlier in a large data set can have a surprisingly large impact on the estimated parameter values.

Section 2.3. Censoring.

A third difficulty faced in empirical implementation of auction models is censoring. Bidding is often a costly activity. In almost all examples of procurement, submitting a bid involves a non-trivial expenditure of managerial effort and other resources.

In applied auction work, not all firms choose to bid on all contracts. Porter (1997) defines the total number of bid opportunities as the number of bidders times the number of contracts. Only 2 percent of bid opportunities actually led to submitted bids in his data set. The set of valuations that bid in equilibrium is censored and will cause difficulty in the estimation of θ.

3. Derivation of the Likelihood.

In order to structurally estimate an auction model, the problems of existence, uniqueness and computation must first be solved. This research exploits recent advances in the computation of game theoretic equilibria to solve these difficulties. (See McKelvy and McLennan (1995) for a survey of recent research.)

McKelvy and Palfrey (1995, 1997) augment standard game theoretic models by introducing errors into agents' decision making process. McKelvy and Palfrey (1995,1997) refer to their solution concept as a quantal response equilibrium. An equilibrium is a distribution over bids, conditional on parameter values. The equilibrium can be used to form a likelihood function for applied work. If the likelihood can be evaluated efficiently, Markov chain Monte Carlo can be used to simulate the posterior distribution of model parameters. In what follows, I introduce the primitives of the game theoretic model and define quantal response equilibrium. The notation and exposition in this section follows McKelvy and Palfrey (1995).

Let $N=\{1,\ldots,n\}$ be the set of bidders in the game. Each agent can choose from a finite set of bids. Agent i's strategy space is $B_i=\{b_{i1},b_{i2},\ldots,b_{iJi}\}$ consisting of J_i possible bids. The set of possible costs for agent i is $C_i=\{c_{i1},\ldots,c_{iTi}\}$, a finite set of T_i possible types. A generic element of B_i is b_i and of C_i is c_i. Define B and C by $B=\Pi_{i\in N}B_i$ and $C=\Pi_{i\in N}C_i$. This is the Cartesian product of all possible bids and types for all the agents in the game. A generic element of B is b and of C is c. Agent i's von Neumann-Morgenstern (vNM) utility function is $u_i:B\times C\rightarrow R$. We will often write $u_i(b_i,b_{-i},c_i,c_{-i})$ to emphasize that agent i's vNM utility depends on i's strategy b_i, the n-1 dimensional vector of strategies of the other players, b_{-i}, player i's type c_i and the vector of types for other players c_{-i}. In a game of incomplete information, c is a random variable with distribution $F(\theta)$, where θ is a vector of parameters. Let Δ_i be the set of probability measures over B_i. A strategy for an agent is an element of Δ_i that is conditioned on the agent's type c_i. In the Bayes-Nash setting, such a strategy will be denoted, at the risk of abusing notation, as $B_i(b_i|c_i)\in\Delta_i$. For all i, b_i in B_i and all c_i in C_i, $B_i(b_i|c_i)\geq 0$. Also, for all c_i, $\sum_{bi} B_i(b_i|c_i)=1$.

Let $F_i(c|\theta)$ denote the marginal distribution of player i's type conditional on θ. Let $F_i(c_{-i}|\theta,c_i)$ be the marginal distribution over c_{-i}, the costs of all firms except i, conditional on c_i and θ. Let $\bar{u}(B_1,\ldots,B_n;c_i,\theta)$ be the expected utility to agent i with type c_i when all other agents bid $B_1(b_1|c_1),\ldots,B_n(b_n|c_n)$. Define $B_{-i}(b_{-i}|c_{-i})=\Pi_{j\neq i}B_j(b_j|c_j)$. Obviously,

$$\bar{u}(B_1,\ldots,B_n;c_i,\theta)=$$
$$\sum_{b\in B}\sum_{c-i\in C-i} u(b_i,b_{-i},c_i,c_{-i})B_i(b_i|c_i)B_{-i}(b_{-i}|c_{-i})F_i(c_{-i}|\theta,c_i).$$

In a Bayes-Nash equilibrium, players maximize their expected utility holding fixed the equilibrium strategies of other firms.

Definition. A **Bayes-Nash** equilibrium is a set of probability measures $B^*_1(b_1|c_1),\ldots,B^*_N(b_N|c_N)$ such that for all i and for all c_i, the probability measure $B^*_i(b_i|c_i)$ maximizes $\bar{u}(B^*_1,\ldots B^*_{i-1},\cdot,B^*_{i+1},\ldots,B^*_n;c_i,\theta)$.

A quantal response equilibrium is a Bayes-Nash equilibrium where players observe $\bar{u}(B_1,\ldots,B_n;c_i,\theta)$ with error. Let b_{ij} refer to the probability measure that puts point mass on the strategy b_{ij}. Then $\bar{u}(b_{ij},B_{-i};c_i,\theta)$ is the expected utility to agent i of bidding b_{ij} given that i's type is c_i, B_{-i} is the vector of strategy vectors for the other agents and θ is the vector of parameters. Define, for all i and all c_i,

$$\hat{u}_{ij}(c_i)=\bar{u}(b_{ij},B_{-i};c_i,\theta)+\varepsilon_{ij}(c_i).$$

Here $\varepsilon_{ij}(c_i)$ is assumed to follow an extreme value distribution . The cdf for the distribution is

$$F(\varepsilon_{ij}(c_i))=\exp(-\exp(-\lambda\varepsilon_{ij}(c_i))).$$

The distribution of the $\varepsilon_{ij}(c_i)$ is independent and identical across the b_{ij} and the c_i. While other distributions could be considered, the extreme value distribution is useful since, as we shall see shortly, it eliminates the need for numerical integration in computing the equilibrium.

Player i, with type c_i will choose the bid b_{ij} that maximizes utility. The bid b_{ij} is chosen if $\hat{u}_{ij}(c_i)\geq\hat{u}_{ik}(c_i)$ for all $k\neq j$. For any real vector $\bar{u}(c_i)\in R^{Ji}$, we can define the ij-best response set $R_{ij}\subseteq R^{Ji}$ as

$$R_{ij}(\bar{u}(c_i))=$$
$$\{\varepsilon_i(c_i)\in R^{Ji}\mid \bar{u}_{ij}(c_i)+\varepsilon_{ij}(c_i)\geq\bar{u}_{ik}(c_i)+\varepsilon_{ik}(c_i)\text{ for all }k=1,\ldots,J_i\}.$$

That is, R_{ij} is the set of errors that make the strategy b_{ij} optimal for agent i. Given a profile of strategies $B_1,\ldots,B_n$ set $R_{ij}(\bar{u}(B_1,\ldots,B_n;c_i))$ equal to the set of errors that will make agent i choose strategy b_{ij} given that other players follow the strategies $B_1,\ldots,B_n$.

Define

$$\sigma_{ij}(\bar{u}(B_1,\ldots,B_n;c_i))=$$
$$\int_{Rij(B_1,\ldots,B_n;c_i)} f(\varepsilon_i(c_i))\,d\varepsilon_i(c_i).$$

The scalar $\sigma_{ij}(\bar{u}(B_1,\ldots,B_n;c_i))$ is the probability that agent i will play strategy b_{ij} when agent i's type is c_i. By well known properties of the extreme value distribution:

$$\sigma_{ij}(\bar{u}(B_1,\ldots,B_n;c_i))=\frac{\exp(\lambda\bar{u}_{ij}(B_1,\ldots,B_n;c_i))}{\sum_{k=1}^{J_i}\exp(\lambda\bar{u}_{ik}(B_1,\ldots,B_n))}.$$

Definition. A **quantal response equilibrium** is a set of strategies B^*_i such that

$$B^*(b_{ij}|c_i)=s_{ij}(\bar{u}\,(B^*_1,\ldots,B^*_n;c_i)).$$

The quantal equilibrium correspondence can be defined as follows:

$B^*(l)=$

$$\{B \in \Delta : B_i(b_{ij} \mid c_i) = \frac{\exp(l\,\bar{u}_{ij}(B_1,\ldots, B_n; c_i))}{\sum_{k=1}^{Ji} \exp(l\,\bar{u}_{ik}(B_1,\ldots, B_n))}\}$$

.

The essential results that McKelvy and Palfrey prove are that

1. A Quantal Response Equilibrium exists.
2. $B^*(l)$ is odd for almost all l.
3. $B^*(l)$ is upper hemi-continuous.
4. The graph of $B^*(l)$ contains a unique branch which starts at the centroid of the simplex, for $l=0$, and converges to a unique Bayes-Nash equilibrium, as l goes to infinity. All other limit points consist of Nash equilibria that are connected to each other in pairs.

The most important result for empirical analysis is point 4 above. A major challenge for empirical game theory is the presence of multiple equilibria. Proposition 4 gives us a unique way to select an equilibrium for a game. For all other solution concepts in games with incomplete information, there will be examples of multiple equilibria that occur for a generic subset of θ. Any econometric procedure must find a sensible way to select among the equilibria. In a quantal response equilibrium every bid will have positive probability. As we will demonstrate below, this is an extremely useful assumption for empirical analysis.

To form the likelihood function used for econometric analysis, one needs to find the probability distribution over bids. The likelihood $f(b_1,\ldots,b_n|q)$ can be formed as

$$f(b_1,\ldots,b_n|q,l)=\sum_{c\in C}\{\prod_{i\in N} B_i(b_i \mid c_i,\, l)\}F(c\mid q)$$

Standard Markov chain Monte Carlo can then be used to simulate the posterior distribution of the model parameters.

4. Comparison of Methods

This section compares the Bayes-Nash equilibrium to the quantal response equilibrium for an asymmetric first price auction. First the bidding functions for the two models are simulated. Second, a computational experiment is proposed to examine the sensitivity of the quantal response and the Bayes-Nash equilibrium to outliers.

To illustrate the difference between the Bayes-Nash equilibrium and the quantal response equilibrium, the distribution of bids under both solution concepts are simulated. Costs were drawn from truncated normal distributions. The auction had 4 firms. The support of the distributions was [0,8]. The mean parameter for the firms are 4.0,4.3,4.5,4.7 and the standard deviation is 1.5. The firms are assumed to be risk neutral. If a firm is the low bidder, it's vNM utility is b_i-c_i, otherwise its utility is zero. These distributions were simulated using 2000 Monte Carlo cost draws. The bidding distributions are shown in figures 1 and 2. Notice that the quantal response model has heavier tails. Also, the bidding distribution of the quantal response model has full support.

[INSERT FIGURES 1 AND 2 HERE]

The following table summarizes the means and standard deviations of the bidding distribution for both the quantal response model and the equilibrium model.

Table 4.1 Comparison of Bidding Models

Parameter	Bayes-Nash	Quantal Response
Mean Bid Firm 1	4.7590	4.9688
Std. Dev. of Firm 1's Bid	1.0471	1.2596
Mean Bid Firm 2	4.8994	5.2116
Std. Dev. of Firm 2's Bid	1.0665	1.2993
Mean Bid Firm 3	5.0742	5.3781
Std. Dev. of Firm 3's Bid	1.0976	1.3031
Mean Bid Firm 4	5.2149	5.4783
Std. Dev. of Firm 4's Bid	1.0960	1.3185

The mean of the quantal response equilibrium is higher than the Bayes-Nash. Intuitively, this comes from the fact that the calculus of equilibrium is extremely delicate for the Bayes-Nash model. Consider the type c_i such that $F_i(c_i|q)=.9$. For this type, the probability that the other three players draw a higher cost is approximately $.1^3$=0.001. In a Bayes-Nash equilibrium, if a firm submits a higher bid, the firm's expected profit conditional on winning is higher. However, the probability that the firm wins is lower. In equilibrium, these marginal benefits and costs are equalized. However, it is easy to see that the marginal benefit and the marginal cost of submitting a higher bid will both be quite small in a neighborhood of the equilibrium.

In the quantal response model, an agent with cost c_1 realizes that it has a very small probability of winning. Since the expected profit for all bids above c_1 is approximately equal, the agent will put positive probability on all of these bids. Therefore, the quantal response model generates heavier tails than the Bayes-Nash model.

Next, a computational experiment is proposed to study the sensitivity of the equilibrium model and the quantal response model to outliers. For simplicity, we use an experiment in which only a single outlier is placed in the fake data set.

The F_i are truncated normal with support [0,8]. There are four firms. The mean parameters for the firms are 4.0, 4.3, 4.5, 4.7 . The F_i has standard deviation parameter 1.5 for all firms. A fake data set with 100 projects was generated for both the quantal response model and the Bayes-Nash model by making random draws out of the F_i and then drawing random bids. A single outlier was placed into both of the fake data sets. The outlier inserted into the quantal response model was 2.2. An outlier of 2.4 was inserted into the Bayes-Nash model. The truncation point, $\underline{b}(q)$ in the Bayes-Nash model is approximately 3.106. In the table below, values from the posterior simulation are reported. The method for simulating the posterior was a normal random walk as described in Bajari (1998). Flat priors for θ were used. The results reported here are for 10,000 draws from the posterior. The value for lambda in the quantal response model was chosen to be 25. In the quantal response model, a grid of 100 bids and 100 costs was chosen. The vNM utility was standard, except that in case of a tie, the payoff to a firm is zero as opposed to alternative tie-breaking rules.

Table 4.2: Bayes-Nash With An Outlier

Parameter	True Value	Posterior Mean	Posterior Std. Dev.
Firm 1's Mean	4.0	3.6908	0.1904
Firm 2's Mean	4.3	3.8550	0.1992
Firm 3's Mean	4.5	4.0600	0.1447
Firm 4's Mean	4.7	4.1715	0.2130
Std. Deviation	1.5	2.1530	0.1069

Table 4.3: Quantal Response With An Outlier

Parameter	True Value	Posterior Mean	Posterior Std. Dev.
Firm 1's Mean	4.0	3.9582	0.1588
Firm 2's Mean	4.3	4.1064	0.1623
Firm 3's Mean	4.5	4.4328	0.1759
Firm 4's Mean	4.7	4.7358	0.1884
Std. Deviation	1.5	1.3373	0.1082

The Bayes-Nash model under-estimates the posterior mean of all the firms. The posterior standard deviation is overestimated. The bid 2.4 must be in the interval $[\underline{b}(\theta), \overline{b}(\theta)]$. In order to decrease $\underline{b}(\theta)$ the model underestimates the mean parameters for the firms and overestimates the standard deviation. The posterior does not exhibit the same sensitivity in the quantal response model.

5. Entry in Auctions.

In most auction data sets, not all bidders bid on all items. For instance, Porter (1997), in his study of bidding for contract to supply school milk, reports that firms only bid in approximately 2 percent of the bid opportunities. This section develops a simple model of entry in auctions that is useful for applied work. This section augments the standard model by assuming that it cost the firm some fixed amount f to take a draw from its cost distribution $F_i(c_i|q)$. The firm does not observe directly the set of participant in the auction. The set of firms N is common knowledge as are the $F_i(c_i)$. The timing in the game is as follows: first firms decide whether or not to bid, second if they choose to bid they draw a cost, third bidding firms submit their bid, not directly observing their competitors. As a solution concept, this section studies the quantal response equilibria of this game. Both the type and strategy space are assumed finite.

The model is solved on all values of a grid with approximately 20,000 different parameter values. All firms were assumed to draw costs from truncated normal distributions with support on the interval [0,8]. There were 3 firms in the game. The means of the firms were allowed to range from 4 to 6. The standard deviation of the firms ranged from .7 to 1.9. The standard deviation for all the firms was assumed to be the same. The entry fee ranged from 0.037 to .2. Table1 reports the results of some simple curve fitting exercises. The mean bid for firm 1, the expected winning bid, the profit for firm 1 and the probability that firm 1 enters are modeled as a polynomial function of the underlying parameters. While describing these relationships as a polynomial is an approximation, the residual between the fitted model and the true values were small on average.

Table 5.1: Bidding in Auctions with Entry

Ind. Variable	Low Bid	Expecte d Bid For Firm 1	Prob. Firm 1 Enters	Profit For Firm 1
intercept	0.9742	3.2742	-3.3496	2.9513
Firm 1's Mean	0.2714	0.4800	1.1753	-1.0049
Firm 1's Mean Squared	0.0026	0.0111	-0.1274	0.0716
Firm 2's Mean	0.2733	-0.1420	0.1625	-0.0362
Firm 2's Mean Squared	0.0024	0.0216	-0.0084	0.0159
Firm 3's				

Mean	0.2712	-0.1420	0.1625	-0.0362
Firm 3's Mean Squared	0.0026	0.0216	-0.0084	0.0159
Std. Dev.	-0.1025	0.3016	0.6361	0.1505
Std. Dev. Squared	-0.0844	-0.1260	-0.1779	0.0019
Entry Fee	0.0584	-0.0334	-0.4903	-0.9845
Entry Fee Squared	2.7576	1.1690	-0.9221	1.1275

A computational experiment is proposed to study the impact of censoring on the auction game. The experiment uses an independent private values model with risk neutral bidders. There are 3 firms in the game. A fake data set with 50 projects was generated. The model with endogenous participation was used to generate the bids. The support of the truncated normals was once again [0,8]. The value of lambda was normalized to 50 for these experiments.

The posterior distributions of two models were simulated using the fake data set. In the first model, the entry decision is included. The first model corresponds to the discussion in the first part of section 5. In the second model, the entry decision is taken as exogenous. The second model is the quantal response model of section 4.

In the Table 5.2, we report the results of simulations of the quantal response model in the case where entry is ignored. In Table 5.3, we report the results of the simulation where entry is explicitly modeled.

Table 5.2: Posterior Simulations Ignoring Entry

Parameter	True Value	Posterior Mean	Posterior Std. Dev.
Firm 1's Mean	4.0	3.6223	0.2652
Firm 2's Mean	4.3	4.3891	0.2072
Firm 3's Mean	4.5	5.0898	0.5627
Standard Dev.	1.0	1.8868	0.0402

Table 5.3: Posterior Simulations for Model with Entry

Parameter	True Value	Post. Mean	Post. Std. Dev.
Firm 1's Mean	4.0	3.9957	0.1107
Firm 2's Mean	4.3	4.2983	0.0967
Firm 3's Mean	4.5	4.5065	0.1132
Standard Dev.	1.0	1.0262	0.0972

If entry is ignored, the mean for firm 1 is underestimated and the mean for firm 3 is over estimated. Also, the standard deviation is almost twice the size as the value that generated the fake data.

Section 6. Conclusions.

This research has discussed limitations in current approaches to structural estimation of auction models. First, existence, uniqueness and characterization theorems for the equilibrium of auction models have only been established under hypotheses that may not be satisfied in practice. Without these basic theorems, it is not possible to derive a likelihood function for econometric analysis. Second, likelihood functions for auction models do not have full support. If an outlier is present in the data set, parameter estimates may be biased. Third, censoring is present in many auction data sets. Since the entry decision is endogenous, failure to model entry will bias parameter estimates.

By using recent advances in computation of game theoretic equilibria, these problems can be avoided. This research computes the bidding functions of the auction game by solving for the quantal response equilibrium, as proposed by McKelvy and Palfrey (1995,1997). In a quantal response model, players observe their equilibrium payoff with some error. The Bayes-Nash equilibrium is a limiting case of the quantal response model. By computing the quantal response equilibrium, it is possible to derive a likelihood function for empirical work.

The quantal response model generates likelihood functions with full support. In computational experiments, the Bayes-Nash model is shown to be very sensitive to outliers while the quantal response model is not. This research numerically studies the equilibrium of an asymmetric auction model with endogenous entry. Equilibrium behavior as a function of the underlying parameters is characterized. A likelihood function for the asymmetric model with entry is derived. In a computational example, it is shown that failure to model entry may bias parameter estimates.

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Figure 1: Bidding Distribution For Bayes−Nash Model

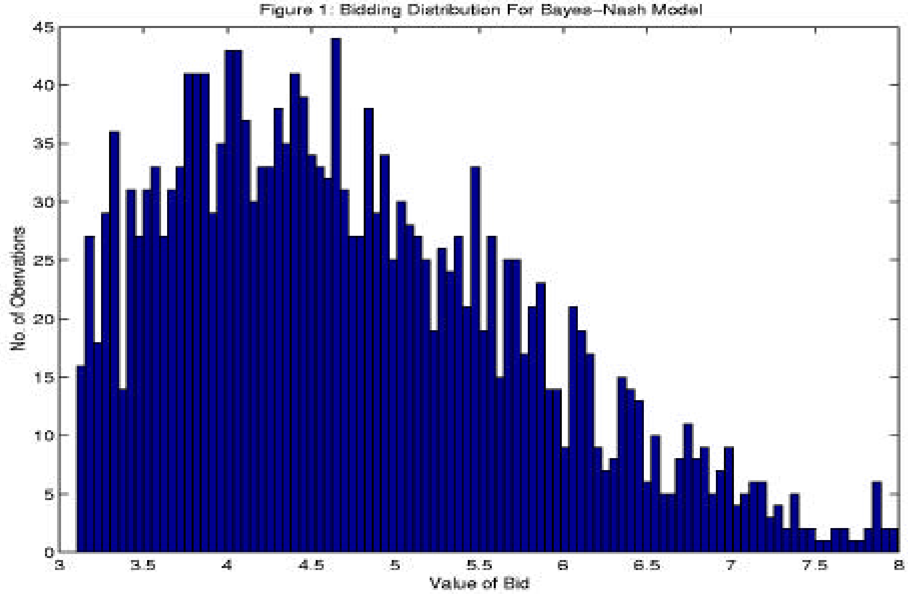

Figure 2: Bidding Distribution For Quantile Response Equilibrium