Toward a Theory of Discounted Repeated Games with Imperfect Monitoring

Dilip Abreu, David Pearce, Ennio Stacchetti, 1990, *Econometrica*

presented by Deniz Çiçek, Ceyhun Elgin and Ayça Özdoğan

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Introduction and Motivation

APS’90

- investigates pure strategy sequential equilibria of a broad class of asymmetric discounted repeated games with imperfect monitoring.

- gives a perspective which views repeated games in terms of a particular intertemporal decomposition using the idea of principal of optimality and dynamic programming.
  - Abreu (1988)
  - Radner, Myerson, Maskin (1986)

- characterizes the equilibrium value set by introducing ”self-generation.”

- analyzes sufficient and necessary conditions to consider only extreme continuation payoffs by introducing ”bang-bang” property.

- presents an algorithm for computation of the payoff set.

- Idea:

  - regard S.E. as specifying a profile of actions for the first period; and a continuation reward function for the remainder of the game.

  - The equilibrium requires that certain incentive constraints to be satisfied: *admissibility*.

- Environment :
  - There is a publicly observable stochastic outcome correlated with players' private choices.
  - None of the player i can observe the actions chosen earlier by other players; nor can i infer this from the signal realizations.

The Model

- **The Stage Game,** G
  - N-players, each player i has a finite strategy set S_i.
  - Payoff function is $\Pi_i : S \rightarrow R$, where $S = S_1 \times ... \times S_N$. (This is an expected value)
  - $\pi_i(p, q_i)$ depend on the realization of a r.v. P and $p \in \Omega \subseteq R$.
  - The distribution of P is $\Psi(.; q)$: the strategy profile q determines the distribution of the public outcome and the support of P is independent of action profile q.
  - $\Pi_i(q) = \int_\Omega \pi_(p, q_i)\Psi(dp; q)$.

- **The Repeated Game,** $G^{\infty}(\delta)$
  - At time t, player i observes p^t and q_i^t; history of the past realizations of r.v. P and history of his own past actions up to time t, respectively.
  - A strategy for player i is a sequence $\sigma_i(t)_{t=1}^{\infty}$, where

    $$\sigma_i(1) \in S_i, \qquad \sigma_i(t) : \Omega^{t-1} \times S_i^{t-1} \rightarrow S_i \quad for \quad t > 1.$$

  - $\sigma|_{p^t,q^t}$: the strategy profile induced by σ after period-t history.
  - In each t, p is drawn independently from $\Psi(.;q)$.
  - Let $v(\sigma) = (v_1(\sigma), ..., v_N(\sigma))$ be the the expected present discounted value induced by σ.

- Assumptions:

  1. S_i is finite, $i = 1, .., N$.
  2. For each $q \in S$, $\Psi(, ; q)$ is absolutely continuous, $g(.; q)$ being the density function.
  3. $\{p \in \Omega | g(p; q) > 0\}$ is independent of q.
     WLOG take $\Omega = \{p | g(p, q) > 0\}$.
  4. $\pi(p, q_i)$ is continuous in p.
  5. G has a NE in pure strategies.

Factorization and Self-Generation

- Let $V \equiv \{v(\sigma)|\sigma \quad is \quad S.E.\}$be the set of S.E.payoffs.

- Consider the maximization problem faced by the player in period 1: Maximizing the sum of current payoffs and expectation of a future reward implicitly ”promised” by σ depending on $p(1)$.

- The reward function, i.e. continuation payoff, must be drawn from V: A S.E. can offer only S.E.rewards.

- **Remark:**
  - Restrict attention to S.E.'s in which each player makes his actions depend only on past signal realizations, not his own previous actions, i.e. to *public strategies*:

$$\sigma_i(t) : \Omega^{t-1} \to S_i \quad for \quad t > 1.$$

  - Focus on $\sigma_i|_{p^t}$ instead of $\sigma_i|_{p^t,q_i^t}$ for all i.
  - Why?
    - Any S.E. σ induces a continuation profile $\sigma|p$ for all $p \in \Omega$.
    - $\sigma|p$ is common knowledge and itself a S.E.
    - Therefore, for all signals p, $v(\sigma|p) \in V$.

- Let $L^{\infty}(\Omega; R^N)$ denote the set of equivalence classes of measurable functions $u : \Omega \to R^N$.

- For any $(q, u) \in S \times L^{\infty}(\Omega; R^N)$,

$$E(q; u) := \delta\{\Pi(q) + \int_{\Omega} u(p)g(p; q)dp\}$$

- For any set $W \subseteq R^N$,
  $L^{\infty}(\Omega; W) \equiv \{u \in L^{\infty}(\Omega; R^N)$ s.t $u(p) \in W$ a.e $p \in \Omega\}$.

- **Definition 1:** For any set W, $(q,u) \in S \times L^{\infty}(\Omega; W)$ is *admissible with respect to W* if

  1. $u(p) \in W$ a.e $p \in \Omega$, and
  2. $E_i(q,u) \geq E_i(q_i', q_{-i}; u)$ for all $q_i' \in S_i$ and $i = 1, ..., N$.

- **Definition 2:** For any set W, $B(W) := \{E(q;u) | (q,u) \quad is \quad admissible \quad w.r.t \quad W\}$.

- **Definition 3:** $W \subseteq R^N$ is self-generating if $W \subseteq B(W)$.

Theorem 1: (*Self-Generation*) For any bounded Borel set $W \subseteq R^N$, if W is self-generating (i.e.$W \subseteq B(W)$), then $B(W) \subseteq V$.

Idea of the proof: By construction. (Assume Ω is countable or finite to ignore measure-theoretic considerations)

- For all $w \in B(W)$, specify S.E. $\widehat{\sigma}$ such that $v(\widehat{\sigma}(w)) = w$.

- Define functions $Q : B(W) \to S$ and $U : B(W) \to L^{\infty}(\Omega, W)$ s.t. $(Q(w), U(w))$ is admissible wrt W with value w, i.e. $E(Q(w), U(w)) = w$ for all $w \in B(W)$.

Theorem 2: (*Factorization*) $V = B(V)$.

Proof: *Idea:* Show that $V \subseteq B(V)$ and that V is a bounded Borel set. Then, by Theorem 1, $V = B(V)$.

- Take $w \in V$ and $\sigma \in SE(G^{\infty}(\delta))$ s.t. $\upsilon(\sigma) = w$, and show that $w \in B(V)$.

- Let (q, u) be s.t. $q := \sigma(1)$ and $u(p) := \upsilon(\sigma|_{p,\sigma(1)})$.

- Then, $w = \upsilon(\sigma) = \delta\{\Pi(\sigma(1)) + \int_{\Omega} \upsilon(\sigma|p, \sigma(1)) g(p; \sigma(1)) dp\} = E(q, u)$

- $\sigma|_{p,\sigma(1)} \in SE(G^\infty)$. Hence, $v(\sigma|p,\sigma(1)) = u(p) \in V$.

- Since σ is a SE and $q = \sigma(1)$, $E_i(q,u) \geq E_i(\gamma_i, q_{-i}, u)$, $\forall \gamma_i \in S_i$.

- Hence, $E(q,u) = w \in B(V)$.

- V is bounded, since $V \subseteq [\delta/(1-\delta) co\{\Pi(q)|q \in S\}]$ and S is finite. V is a Borel set by Theorem 4.

- Thus, $V \subseteq B(V) \subseteq V$ implies $B(V) = V$.

Bang-Bang Reward Functions and the Structure of Equilibria

- An *extreme point* of $W \subset R^N$ is a point that is not a convex combination of other points in W.

  $extW$ denote the set of extreme points in W.

Definition 4: u has the *bang-bang property* if $u(p) \in extW$ a.e $p \in \Omega$.

Theorem 3: Let $W \subseteq R^N$ be compact and $(q, \hat{u})$ be an admissible pair with respect to coW. Then there exists a function $\bar{u} \in L^\infty(\Omega; extW)$ such that $(q, \bar{u})$ is admissible with respect to W and $E(q; \bar{u}) = E(q; \hat{u})$.

Corollary: Let $W \subseteq R^N$ be compact. Then $B(W) = B(coW)$.

Lemma 1: The operator B satisfies the following properties:

1. if $W \subseteq W' \subseteq R^N$, then $B(W) \subseteq B(W')$
2. if $W \subseteq R^N$ is compact, $B(W)$ is compact.

Theorem 4: V is compact.

Proof:

- Recall from the proof of Theorem 2 that V is bounded and self-generating. Since V is bounded, $cl(V)$ is compact.
- By monotonicity, $V = B(V) \subseteq B(cl(V))$, and by lemma 1, $B(cl(V))$ is compact. Hence, $cl(V) \subseteq B(cl(V))$.
- By self-generation, $cl(V) \subseteq V$. Thus, V is closed and compact.

Computation and Monotonicity in δ

- Analogous to Howard's value-iteration procedure

- An algorithm is presented to find V, the biggest bounded fixed point of B

- Start with a compact set W_0 which satisfies $V \subseteq B(W_0) \subseteq W_0$

- Proceed by computing the monotonically decreasing sequence of compact sets $W_n := B(W_{n-1})$,$n = 1, 2...$

- The limit of this process yields $V = \lim_{n\to\infty} W_n$=$\cap_1^\infty W_n$

Idea of the proof:

- $\{W_n\}$ is decreasing and W_n contains V for all n.

- Therefore, $V \subseteq W_\infty$

- How do we know that it converges to V but not to a larger set, W_∞?

- By self-generation and corollary to theorem 3 it suffices to show that $W_\infty \subseteq B(coW_\infty)$

- Consider any w in W_∞. By definition, for all n there is a pair (q_n, u_n) admissible with respect to W_n such that $E(q^n, u^n) = w$.

- The sequence $\{q_n, u_n\}$ must have a convergent subsequence (*proven in the article*) and the limit pair $\{q, u\}$ will also have expected value w and is admissible w.r.t coW_∞ (*proven in the article*)

- Thus,W_∞ is self-generated, so $W_\infty \subseteq B(W_\infty) \subseteq V$.

Theorem 6: (*Monotonicity in Discount Factor*) Let δ_1 and δ_2 be two discount factors such that $0 < \delta_1 < \delta_2 < 1$.
Then $[(1-\delta_1)/\delta_1]V(\delta_1) \subseteq [(1-\delta_2)/\delta_2]V(\delta_2)$

Note: Intuitively this says that if the discount factor increases from δ_1 to δ_2 and payoffs are appropriately normalized, the original set of equilibrium values is contained in the new set of values associated with δ_2

Conclusion

- APS’90 investigates pure strategy sequential equilibria of a broad class of asymmetric discounted repeated games with imperfect monitoring.
- Characterization of the equilibrium value set by using dynamic programming for repeated discounted games with imperfect monitoring.
- This paper presents an algorithm for computation of the payoff set.
- Some results are also applicable to games with perfect monitoring.

- Hybrid cases (i.e. models between perfect monitoring and those having a publicly observed random signal with constant support, are not covered.)

- Mixed strategy equilibria of repeated discounted games are not examined.