General Competitive Analysis in an Economy with Private Information

Prescott & Townsend (1984 IER)

Elisa Belfiori, Manuel Macera & Hitoshi Tsujiyama

September 2008

Purpose of the paper

- Goal: Extend Competitive General Equilibrium Theory to allow for private information
- Challenges
  - Overcome problems arising from IC contraints
  - Establish Existence of Pareto Optimal and CE allocations
  - Welfare Theorems.

Plan of the presentation

1. Environment
2. Economy
3. Existence of PO
4. Existence of CE
5. Welfare theorems

Environment

Description

- T period economy
- Continuum of agents with private information $\theta \in \Theta$ where $\Theta = \{\theta^1, ..., \theta^n\}$: preference shocks at the beginning of the period drawn from $\lambda : \Theta \rightarrow [0, 1]$
- Realizations are time independent and identically distributed across agents
- l commodities
- Endowment $e_t \geq 0, \forall t = 0, 1, ...T$

Environment

Description

- Let c_t be a non negative consumption vector
- Preferences are represented by

$$E_0\left(\sum_{t=0}^{T} U\left(c_t, \theta_t\right)\right)$$

where U is continuous, concave and strictly increasing in the first argument.

Environment

Two problems arise in this environment with private information

1. Shock contingent allocations may not be optimal (Example 1)
2. When we introduce IC constraints, nonconvexities arise (Example 2)

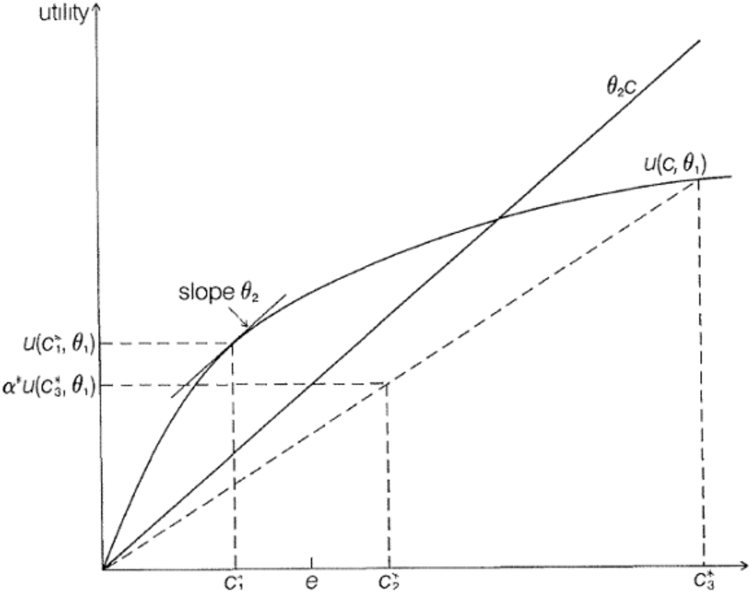
Environment

Example 1

- Let $T = 1$, $I = 1$, $\Theta = \{\theta_1, \theta_2\}$, $e_0 = 0$, $e_1 = e$
- $U : \mathbb{R}_+ \times \Theta \rightarrow \mathbb{R}$.Let $U'(\infty, \theta_1) = 0$, $U(c, \theta_2) = \theta_2 c$. Let $U'(e, \theta_1) < \theta_2$.
- Full information PO solves

$$\max_{c_1, c_2} \lambda(\theta_1) U(c_1, \theta_1) + \lambda(\theta_2) \theta_2 c_2$$
$$s.t. \quad \lambda(\theta_1) c_1 + \lambda(\theta_2) c_2 = e$$

- Thus (c_1^*, c_2^*) is such that $U'(c_1^*, \theta_1) = \theta_2$, $\lambda(\theta_1) c_1^* + \lambda(\theta_2) c_2^* = e$. But this is unattainable with private information!
- Private information PO should add IC. But then, the unique implementable allocation is $c_1 = c_2 = e$



utility
$\theta_2 c$
$u(c, \theta_1)$
slope θ_2
$u(c_1^*, \theta_1)$
$\alpha^* u(c_3^*, \theta_1)$
c_1^*
e
c_2^*
c_3^*

Environment

Example 2

- Let $I = 2,\ T = 1, \Theta = \{\theta', \theta''\}, e_0 = 0$
- Incentive Compatible Constraints

$$U(c(\theta'), \theta') \geq U(c(\theta''), \theta') \quad (1)$$

$$U(c(\theta''), \theta'') \geq U(c(\theta'), \theta'') \quad (2)$$

- Consider allocations $\{c_A(\theta'), c_A(\theta'')\}$ and $\{c_B(\theta'), c_B(\theta'')\}$ satisfying (1) above with equality and $c_A(\theta'') \neq c_B(\theta'')$.

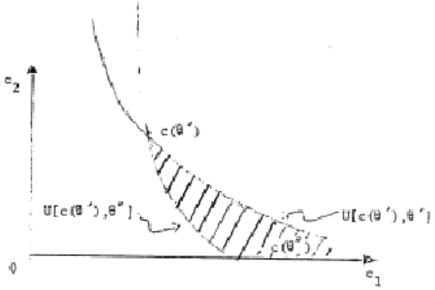
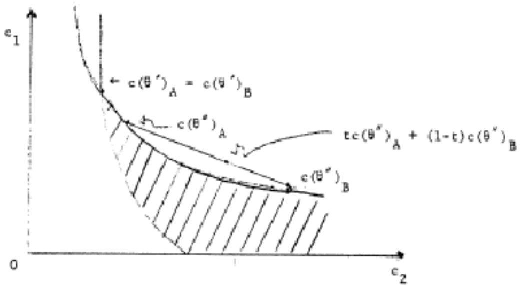


Figure 2a



Economy

Example 1 again

- Recall the Example 1
- Consider the lotteries
  $\mu_1 = (c_1^*, 1; 0, 0), \mu_2 = (0, 1 - \alpha^*; c_3^*, \alpha^*)$
  s.t.$E(\mu_2) = c_2^*$.
- By construction this $\mu = (\mu_1, \mu_2)$ satisfies RC's and IC's:

$$
\begin{aligned}
EU(\mu_{1,}\theta_1) &\geq EU(\mu_{2,}\theta_1) \\
EU(\mu_{2,}\theta_2) &\geq EU(\mu_{1,}\theta_2)
\end{aligned}
$$

- Moreover this lottery attains the same utility as the optimal allocation with full information. Hence it is private information PO.
- This reveals the necessity of lotteries as an allocation device.

Economy

Commodity Space

- Finite Horizon: $T = 2$.
- Commodity Space
  - Underlying Consumption Possibility Set

$$C = \left\{ c \in \mathbb{R}^l : 0 \leq c \leq b \right\} \text{ for some } b \in \mathbb{R}^l$$

  - Let $\mathcal{C}$ be the set of Borel subsets of C. Then

$$\mathcal{L} = \left\{ \left(\mu_0, \{\mu_1(\theta_1)\}_{\theta_1 \in \Theta}, \{\mu_2(\theta_1, \theta_2)\}_{(\theta_1, \theta_2) \in \Theta} \right) \text{ s.t. } \mu : \mathcal{C} \to \mathbb{R} \right\}$$

    is the commodity space. A typical element of $\mathcal{L}$ is a $1 + n + n^2$ tuple of set functions on $\mathcal{C}$

Economy

Consumption set

- Let $P \equiv \left\{x \in \mathcal{L} : x \text{ is a } 1+n+n^2 \text{ tuple of probability measures}\right\}$
- Let $X \subset \mathcal{L}$ be the consumption set

$$X = \{x \in P : x \text{ satisfies IC1 and IC2}\}$$

where

$$\int U(c,\theta_2)x_2(dc,\theta_1,\theta_2) \geq \int U(c,\theta_2)x_2(dc,\theta_1,\theta_2'), \forall\ \theta_1,\theta_2,\theta_2' \in \Theta \quad \text{(IC2)}$$

$$\int U(c,\theta_1)x_1(dc,\theta_1) + \sum_{\theta_2} \lambda(\theta_2) \int U(c,\theta_2)x_2(dc,\theta_1,\theta_2)$$

$$\geq \int U(c,\theta_1)x_1(dc,\theta_1') + \sum_{\theta_2} \lambda(\theta_2) \int U(c,\theta_2)x_2(dc,\theta_1',\theta_2),\ \forall\ \theta_1,\theta_1' \in \Theta \quad \text{(IC1)}$$

Economy

Preferences and Endowments

- Expected Utility of type i (i.e. $\theta_0 = \theta^i$)

$$
\begin{aligned}
W(x,i) &= \int U(c,i)x_0(dc) + \sum_{\theta_1}\lambda(\theta_1)\int U(c,\theta_1)x_1(dc,\theta_1) \\
&+ \sum_{\theta_1}\lambda(\theta_1)\sum_{\theta_2}\lambda(\theta_2)\int U(c,\theta_2)x_2(dc,\theta_1,\theta_2)
\end{aligned}
$$

- Let $\xi \in P$

$$
\xi_0\left(e_0, i\right) = 1, \forall i
$$
$$
\xi_1\left(e_1, \theta_1\right) = 1, \forall \theta_1 \in \Theta
$$
$$
\xi_2\left(e_2, \theta_1, \theta_2\right) = 1, \forall \theta_1, \theta_2 \in \Theta
$$

Economy

Definition

A pure exchange economy $\mathcal{E}$ is the following collection:

$$\mathcal{E} = \left\{ \left\{ \lambda\left(i\right) \right\}_{i \in \Theta}, \mathcal{L}, X, \left\{ W\left(\cdot, i\right) \right\}_{i \in \Theta}, \xi \right\}$$

Existence of Pareto Optima

Implementable Allocation

An implementable allocation for this economy is a collection $\{x_i\}_{i \in \Theta}$, $x_i \in X$ such that:

$$\sum_i \lambda(i) \int c x_{i0}(dc) \leq e_0 \tag{RC 0}$$

$$\sum_i \lambda(i) \sum_{\theta_1 \in \Theta} \lambda(\theta_1) \int c x_{i1}(dc, \theta_1) \leq e_1 \tag{RC 1}$$

$$\sum_i \lambda(i) \sum_{\theta_1 \in \Theta} \lambda(\theta_1) \sum_{\theta_2 \in \Theta} \lambda(\theta_2) \int c x_{i2}(dc, \theta_1, \theta_2) \leq e_2 \tag{RC 2}$$

$$W(x_i, i) \geq W(x_j, i), \forall i, j \in \Theta \tag{PSSC}$$

Existence of Pareto Optima

Pareto Optimal Allocation

An implementable allocation $\{x_i\}_{i\in\Theta}$ is Pareto Optimal if there is no implementable allocation $\{\hat{x}_i\}_{i\in\Theta}$such that

$$W(\hat{x}_i, i) \geq W(x_i, i)$$

with strict inequality for some i

Existence of Pareto Optima

- The SPP for a private information economy is

$$\max_{\{x_i\}_{i\in\Theta}} \sum_i w(i)\, W(x_i, i)$$

$$\begin{aligned} s.t. \quad & x_i \in X \\ & RC0 - RC2 \\ & PSSC \end{aligned}$$

- Outline of the proof: Since C is compact under the appropiate topology, P is compact. The constraint set is closed and thus compact. Since $W(\cdot, i)$ is continuous $\forall i$, a maximum is guaranteed to exist. Hence PO exist.

Existence of CE

Introducing a firm

- Let

$$Y = \left\{ \ y \in \mathcal{L} : \ \text{ satisfying (1) (2) and (3) below } \right\}$$

$$\int c y_0 \left(dc\right) \leq 0 \tag{1}$$

$$\sum_{\theta_1 \in \Theta} \lambda\left(\theta_1\right) \int c y_1 \left(dc, \theta_1\right) \leq 0 \tag{2}$$

$$\sum_{\theta_1 \in \Theta} \lambda\left(\theta_1\right) \sum_{\theta_2 \in \Theta} \lambda\left(\theta_2\right) \int c y_2 \left(dc, \theta_1, \theta_2\right) \leq 0 \tag{3}$$

  be the production set
- A firm is an intermediary
- An element of Y is a signed measure.

Existence of CE

- A state for this economy is a $n + 1$-tuple $\left[\{x_i\}_{i\in\Theta}, y\right]$ of elements of $\mathcal{L}$
- A state is attainable iff $x_i \in X, \forall i$, $y \in Y$, $\sum_i \lambda(i)\, x_i - y = \xi$
- A price system is a linear functional $v : \mathcal{L} \to \mathbb{R}$

Existence of CE

Definition

A CE is a state $\left[\{x_i^*\}_{i\in\Theta}, y^*\right]$ and a price system , v^* such that

1. For all i, x_i^* maximizes $W(x_i, i)$ s.t. $v^*(x_i) \leq v^*(\xi)$, $x_i \in X$
2. y^* maximizes $v^*(y)$ s.t. $y \in Y$
3. $\sum_i \lambda(i)\, x_i^* - y^* = \xi$

Existence of CE

Outline of the proof

1. Construct a sequence of restricted economies. For each k:
   - Restrict C to a finite number of points: C^k
   - Let L^k be the finite dimensional subspace of L
   - Define $X^k = X \cap L^k$, $Y^k = Y \cap L^k$, and $\xi^k = \xi \in L^k$
2. CE in the k^{th} restricted economy can be proved to exist following Debreu (1962)
3. Let the grid get finer and construct a sequence of CE.
4. Compactness of C ensures that this sequence has an accumulation point, which can be shown to be the CE for the unrestricted economy.

Welfare Theorems

- A state $\left[\{x_i\}_{i\in\Theta}, y\right]$ is PO if it is attainable, satisfies PSSC and $\nexists$ attainable $\left[\{\hat{x}_i\}_{i\in\Theta}, \hat{y}\right]$ satisfying PSSC and $W(\hat{x}_i, i) \geq W(x_i, i)$ $\forall i$ with $>$ for some j.

Theorem (First Welfare Theorem)

Let $\left[\{x_i^*\}_{i\in\Theta}, y^*\right]$ and v^*be a CE. Assume that for every i, x_i^* is not a satiation point. Then $\left[\{x_i^*\}_{i\in\Theta}, y^*\right]$ is a PO allocation.

Standard Proof.

- The Second Welfare Theorem does not hold in the usual sense.
- However, any PO allocation can be supported as a kind of CE.

Example 1 (continued)

Can the optimal allocation be supported as a CE?

- Recall the Example 1
- The lottery $x_1 = (c_1^*, 1; 0, 0), x_2 = (0, 1 - \alpha^*; c_3^*, \alpha^*)$ is a PO with private information.
- Can we support it with a CE?
- In CE, HH buys an insurance contract $\{x(c, \theta), c \in C, \theta \in \Theta\}$ in the planning period.
- HH receives c with prob $x(c, \theta)$ based on its announcement in the consumption period.
- A price system $\{p(c, \theta)\}$ determines the cost of a contract.
- Endowment $\xi(e, \theta) = 1 \; \forall \theta \in \Theta$

Example 1 (continued)

Competitive Equilibrium

A competitve equilibrium is a price system $\{p(c,\theta)\}$ and an allocation $\{x^*, y^*\}$ s.t.

1. x^* solves HHP

$$\max_{x \in X} \sum_{\theta} \lambda(\theta) \sum_{c} x(c,\theta) U(c,\theta) \tag{HHP}$$

$$s.t \quad \textstyle\sum_{\theta} \sum_{c} p^*(c,\theta) x(c,\theta) \leq \sum_{\theta} \sum_{c} p^*(c,\theta) \xi(c,\theta)$$

2. y^* solves FP

$$\max_{y} \sum_{\theta} \sum_{c} p^*(c,\theta) y(c,\theta) \tag{FP}$$

$$s.t. \quad y \in Y \equiv \{y(c,\theta) : c \in C, \theta \in \Theta, \textstyle\sum_{\theta} \lambda(\theta) \sum_{c} c y(c,\theta) \leq 0\}$$

3. $x^* - y^* = \xi$

Example 1 (continued)

Competitive Equilibrium

- Consider a price system $p^*(c,\theta) = \lambda(\theta)c$
- And an allocation $\{x^*, y^*\}$ such that

$$x^*(c,\theta) = \begin{cases} 1, & \textit{if } c = c_1^* \textit{ and } \theta = \theta_1 \\ \alpha^* & \textit{if } c = \frac{c_2^*}{\alpha^*} \textit{ and } \theta = \theta_2 \\ 1-\alpha^* & \textit{if } c = 0 \textit{ and } \theta = \theta_2 \\ 0 & \textit{otherwise} \end{cases}$$

$$y^* = x^* - \xi$$

- By construction, the allocation and the price system solves for the CE and is feasible.
- Then it is a CE that supports the private information PO allocation x^*.