Inequality and Social Discounting

Farhi & Werning, JPE 2007

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Introduction

- Balancing act between equality of opportunity and incentives for altruistic parents.
- Atkeson and Lucas (REStud, 1992): Immiseration result.
- F&W modify welfare criterion: Desirability of insuring the unborn against ancestors' luck.
- Main result: set social discount factor higher than individual's to avoid immiseration and allow social mobility.
- Closely related: Phelan (REStud, 2006)

A simple deterministic example

Social Discounting

- Two periods: 1) Parent 2) Child.
- Child's utility: $v_1 = U(c_1)$.
  Parent's utility: $v_0 = U(c_0) + \beta v_1 = U(c_0) + \beta U(c_1)$.
- Welfare criterion:

$$W \equiv v_0 + \alpha v_1 = U(c_0) + (\alpha + \beta) U(c_1) = U(c_0) + \hat{\beta} U(c_1).$$

- $\hat{\beta} \equiv$ social discount factor.
- $\alpha = 0 \Rightarrow \beta = \hat{\beta}$ (Atkeson and Lucas)

A simple deterministic example

Planning Problem

- Two dynasties: A, B
- Planner must divide fixed endowment $2e$ between $c_{A,t}$ and $c_{B,t} = 2e - c_{A,t}$.
- Government commits to difference Δ in welfare between dynasties.
- Planner's problem:

$$\max_{c_{A,t}} \sum_{t=0}^{\infty} \hat{\beta}^t \left[\frac{1}{2} U(c_{A,t}) + \frac{1}{2} U(2e - c_{A,t}) \right]$$

$$\text{s.t.} \sum_{t=0}^{\infty} \beta^t U(c_{A,t}) - \sum_{t=0}^{\infty} \beta^t U(2e - c_{A,t}) = \Delta$$

A simple deterministic example

Imperfect Inheritability

- FOC for interior optimum:

$$\frac{U'(c_{A,t})}{U'(c_{B,t})} = \frac{1-\lambda(\beta/\hat{\beta})^t}{1+\lambda(\beta/\hat{\beta})^t}, \quad t = 0, 1, \ldots$$

- $\Delta > 0 \Rightarrow \lambda > 0 \Rightarrow c_{A,t} > c_{B,t}$.
- $\hat{\beta} = \beta \Rightarrow$ ratio constant, differences persist forever.
- $\hat{\beta} > \beta \Rightarrow$ consumption differences shrink over time.
- Inequality Δ exogenous. In model with shocks inequality generated to provide incentives!

A simple deterministic example

Consumption paths

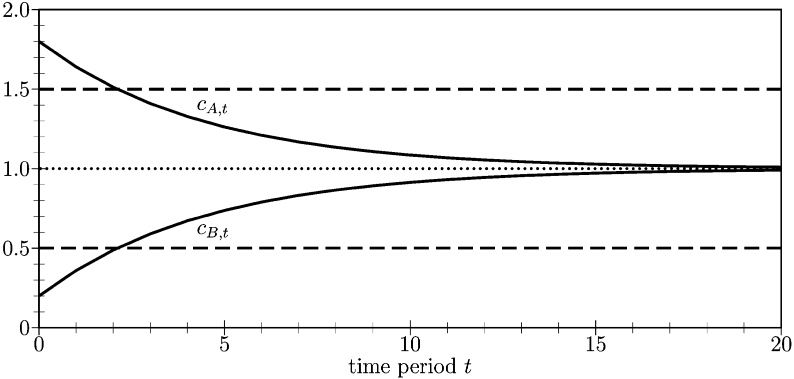

FIG. 1.—Consumption paths for groups *A* and *B*. Solid lines represent the case with $\hat{\beta} > \beta$; the dotted line at $c = e = 1$ is the steady state. The horizontal dashed lines represent the Atkeson-Lucas case with $\hat{\beta} = \beta$.

Intergenerational Insurance Problem

Economy Setup

- Continuum of one-period lived individuals, identical preferences. Replaced by single descendant next period.
- Welfare satisfies: $v_t = E_{t-1}[\theta_t U(c_t) + \beta v_{t+1}]$
- Private taste shock: $\theta \in \Theta$ iid. Θ finite.
- Preference over entire dynasty ("welfare"):

$$v_t = \sum_{s=0}^{\infty} \beta^s E_{t-1}[\theta_{t+s} U(c_{t+s})]$$

- Aggregate endowment e.
- Dynasty's reporting strategy: $\sigma \equiv \{\sigma_t\}_{t=0}^{\infty}$, $\sigma_t : \Theta^{n+1} \longrightarrow \Theta$.
- Dynasty's founder entitled to welfare $v \sim \psi$.

Intergenerational Insurance Problem

Feasible Allocations

- Let $u_t(\theta^t) \equiv U(c(\theta^t))$. $C \equiv U^{-1}$.
- An allocation is a sequence $\{u_t^v\}$ $\forall$ v.
- Given e, ψ *feasibility* requires:

$$\sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^{t+1}} \beta^t \theta_t u_t^v(\theta^t) Pr(\theta^t) = v, \tag{PK}$$

$$\sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^{t+1}} \beta^t \theta_t [u_t^v(\theta^t) - u_t^v(\sigma^t(\theta^t))] Pr(\theta^t) \geq 0, \ \forall \ \sigma, \tag{IC}$$

$$\int \sum_{\theta^t \in \Theta^{t+1}} C(u_t^v(\theta^t)) Pr(\theta^t) d\psi(v) \leq e, \quad t = 0, 1, \ldots \tag{RF}$$

Intergenerational Insurance Problem

Social Discounting

- Suppose planner wishes to maximize the welfare criterion:

$$\begin{aligned} W &= \sum_{t=1}^{\infty} \alpha_t E_{t-1} v_t,\ \alpha_t \geq 0\ \forall\ t \\ &= \sum_{t=1}^{\infty} \delta_t E_{t-1}[\theta_t U(c_t)],\ \delta_t \equiv \sum_{s=0}^{t} \beta^{t-s} \alpha_s. \end{aligned}$$

- Social preferences more patient: $\dfrac{\delta_{t+1}}{\delta_t} = \beta + \dfrac{\alpha_{t+1}}{\delta_t} > \beta$.
- Letting $\alpha_t = \hat{\beta}^t$, $\hat{\beta} > \beta$:

$$\sum_{t=1}^{\infty} \alpha_t E_{t-1} v_t = \frac{1}{\hat{\beta} - \beta} \left\{ \sum_{t=0}^{\infty} \hat{\beta}^t E_{t-1}[\theta_t U(c_t)] - v_0 \right\}$$

Intergenerational Insurance Problem

The Planner's Problem

The social planning problem (SPP) is:

$$S(\psi; e) \equiv \sup_{\{u_t^v\}} \int \sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^{t+1}} \hat{\beta}^t \theta_t u_t^v(\theta^t) Pr(\theta^t) d\psi(v)$$

subject to (PK),(IC),(RF).

Intergenerational Insurance Problem

Steady States

- Social planning problem is recursive with state variable ψ_t.
- $\psi_{t+1} = \Psi \psi_t$.
- Steady state is a fixed point: $\psi^* = \Psi \psi^*$.
- Direct approach using entire distribution ψ_t as state variable is intractable.
- Alternative approach: Convert to multiple subproblems with one-dimensional state variables. . .

A Bellman Equation

The Relaxed Planning Problem

- Define a *relaxed planning prob. (RPP)* by replacing (RF) with:

$$\int \sum_{t=0}^{\infty} Q_t \sum_{\theta^t \in \Theta^{t+1}} C\left(u_t^v\left(\theta^t\right)\right) p\left(\theta^t\right) d\psi\left(v\right) \leq e \sum_{t=0}^{\infty} Q_t$$

  for some positive sequence $\{Q_t\}$
- Any solution to RPP satisfying (RF) is also a solution to SPP.
- Then any steady-state solution to RPP is a steady-state solution to SPP.
- Steady-state requires $Q_t = \hat{\beta}^t$, hence we can write:

$$\int \sum_{t=0}^{\infty} \hat{\beta}^t \sum_{\theta^t \in \Theta^t} C\left(u_t^v\left(\theta^t\right)\right) p\left(\theta^t\right) d\psi\left(v\right) \leq e \sum_{t=0}^{\infty} \hat{\beta}^t \quad \text{(IRC)}$$

A Bellman Equation

The Component Planning Problem

- Let η be the multiplier on the (IRC) constraint. Define:

$$\mathcal{L}^v = \sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^{t+1}} \hat{\beta}^t \left[\theta_t u_t^v \left(\theta^t\right) - \eta C \left(u_t^v \left(\theta^t\right)\right)\right] p \left(\theta^t\right) \quad \text{(LV)}$$

- Then, the RPP solves:

$$\max_{\{u_t^v\}} \mathcal{L} = \int \mathcal{L}^v d\psi \left(v\right) \quad s.t. \quad \text{(PK), (IC)}$$

- Maximizing $\mathcal{L}$ is equivalent to pointwise maximization of $\mathcal{L}^v$ s.t. (PK), (IC). Call this *component planning prob. (CPP)*.
- For any e there exists $\eta > 0$ such that an allocation $\{u_t^v\}$ solves RPP iff for each v the allocation solves CPP given η, e.

A Bellman Equation

Two Results

Define $k(v) \equiv \sup_{\{u_t^v\}} \mathcal{L}^v$ s.t. (PK), (IC).

Theorem

$k\left(v\right)$ is strictly concave, continuous, and differentiable. Moreover $\lim_{v\to\underline{v}} k'\left(v\right)=\infty$ and $\lim_{v\to\overline{v}} k'\left(v\right)=-\infty$.

Theorem

The value function of CPP satisfies the Bellman equation:

$$k\left(v\right)=\max_{u,w} E\left[\theta u\left(\theta\right)-\eta C\left(u\left(\theta\right)\right)+\hat{\beta}k\left(w\left(\theta\right)\right)\right] \tag{BE}$$

$$s.t. \quad v=E\left[\theta u\left(\theta\right)+\beta w\left(\theta\right)\right] \tag{PK'}$$

$$\theta u\left(\theta\right)+\beta w\left(\theta\right)\geq\theta u\left(\theta'\right)+\beta w\left(\theta'\right)\ \ \forall\ \theta,\theta'\in\Theta \tag{IC'}$$

Mean Reversion

Let multipliers on (PK') and (IC') be λ and $\mu(\theta, \theta')$. Then, FOCs read:

$$\left[\theta - \eta C'(u(\theta)) - \theta\lambda\right] p(\theta) + \sum_{\theta' \in \Theta} \left[\theta\mu\left(\theta, \theta'\right) - \mu\left(\theta', \theta\right)\right] = 0 \quad (1)$$

$$[\hat{\beta} k'(w(\theta)) - \beta\lambda] p(\theta) + \beta \sum_{\theta' \in \Theta} \left[\mu\left(\theta, \theta'\right) - \mu\left(\theta', \theta\right)\right] = 0 \quad (2)$$

while the envelope condition implies $k'(v) = \lambda$.

Mean Reversion

- Summing (2) over θ, we obtain:

  $$\sum_{\theta\in\Theta} k'\left(w\left(\theta\right)\right) p\left(\theta\right) = \frac{\beta}{\hat{\beta}} k'\left(v\right) \tag{3}$$

  which states that $\left\{k'\left(v\right)\right\}$ is a supermartingale. By the supermartingale convergence theorem (Doob, 1953), we know the series $\left\{k'\left(v_t\right)\right\}$ will converge almost surely towards zero.
- By the previous theorem then we prove that $k\left(v\right)$ has an interior maximum at $v^* > \underline{v}$. NO IMMISERATION

Existence of Steady State

Rewrite (3) as:

$$E_{t-1}\left[1-k'\left(v_{t+1}\right)\right]=\frac{\beta}{\hat{\beta}}\left[1-k'\left(v_{t}\right)\right]+1-\frac{\beta}{\hat{\beta}} \tag{4}$$

Theorem

For $1-k'(v)\geq 0$,

$$\underline{\gamma}\left[1-k'(v)\right]+1-\frac{\beta}{\hat{\beta}}\leq 1-k'\left(v'\right)\leq\overline{\gamma}\left[1-k'(v)\right]+1-\frac{\beta}{\hat{\beta}}$$

for some $\underline{\gamma}$ *and* $\overline{\gamma}$ *with* $\underline{\gamma}\leq\beta/\hat{\beta}\leq\overline{\gamma}$ *and* $\underline{\gamma},\overline{\gamma}\rightarrow\beta/\hat{\beta}$ *as* $\overline{\theta}/\underline{\theta}=\max\Theta/\min\Theta\rightarrow 1$.

Example 1

No uncertainty

- Suppose $\bar{\theta} = \underline{\theta}$, then by the previous theorem:

$$\underline{\gamma} = \bar{\gamma} = \frac{\beta}{\hat{\beta}}.$$

and

$$1 - k'(v') = 1 - \frac{\beta}{\hat{\beta}} k'(v).$$

- Therefore $k'(v) \to 0$.

Example 2

Convergence to stable distribution

- Suppose $k'(v) = 1$, then by the previous theorem:

$$1 - k'(v') = 1 - \frac{\beta}{\hat{\beta}}.$$

- By concavity of $k(v)$, $v' > v$.
- Next period:

$$\left[1 - \frac{\beta}{\hat{\beta}}\right]\left[1 + \underline{\gamma}\right] \leq 1 - k'\left(v''\right) \leq \left[1 - \frac{\beta}{\hat{\beta}}\right]\left[1 + \overline{\gamma}\right]$$

- By concavity of $k(v)$, $v'' > v'$ whenever $\underline{\gamma} > 0$.

Existence of Steady State

The existence of the steady-state is guaranteed by the following:

Theorem

The Markov process $\{v_t\}$ *implied by the solution has an invariant distribution* ψ^* *with no mass at misery* $\psi^*(\underline{v}) = 0$ *and* $\int k'(v)\, d\psi^*(v) = 0$ *if any of the following is satisfied: utility is bounded below, utility is bounded above,* $\overline{\gamma} < 1$ *or* $\underline{\gamma} > 0$*.*

Mean Reversion

- By summing (1) over θ, we obtain:

$$\eta \sum_{\theta \in \Theta} C'\left(u\left(\theta\right)\right) p\left(\theta\right) = 1 - k'\left(v\right) \tag{5}$$

- By putting together (5) and (4) we finally derive:

$$E_{t-1}\left[\frac{1}{U'\left(c_{t+1}\right)}\right] = \frac{\beta}{\hat{\beta}} E_{t-1}\left[\frac{1}{U'\left(c_{t}\right)}\right] + \frac{1}{\eta}\left(1 - \frac{\beta}{\hat{\beta}}\right) \tag{6}$$

Example

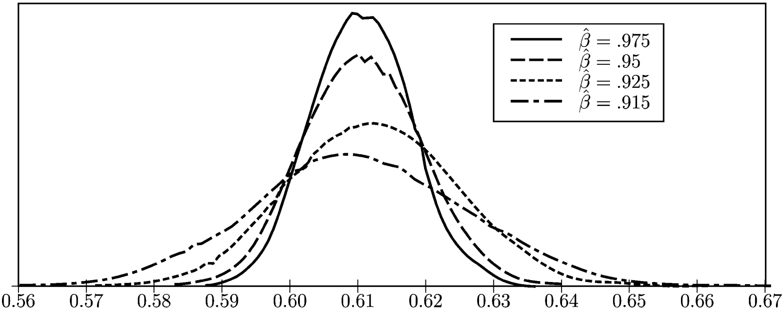

FIG. 2.—Invariant distributions for welfare, measured in consumption-equivalent units $C((1-\beta)v)$, for various values of $\hat{\beta}$.

A Bellman Equation

Proof of theorem

By definition of supremum:

$$k(v) \leq \max_{u,w} \quad E\left[\theta u(\theta) - \eta C(u(\theta)) + \hat{\beta} k(w(\theta))\right]$$

$$\text{s.t.} \quad (\text{PK}), (\text{IC})$$

At the same time, for every v and $\varepsilon > 0$ there exists a plan $\left\{\widetilde{u}_t\left(\theta^t; v, \varepsilon\right)\right\}$ that is IC and delivers v with value:

$$\sum_{t=0}^{\infty} \hat{\beta}^t E_{-1}\left[\theta_t \widetilde{u}_t\left(\theta^t; v, \varepsilon\right) - \eta C\left(\widetilde{u}_t\left(\theta^t; v, \varepsilon\right)\right)\right] \geq k(v) - \varepsilon$$

Let

$$\left(u_t^*(\theta), w^*(\theta)\right) \in \arg\max_{u,w} E\left[\theta u(\theta) - \eta C(u(\theta)) + \hat{\beta} k(w(\theta))\right]$$

A Bellman Equation

Proof of theorem (cont.)

Consider the plan providing $u_0(\theta_0) = u^*(\theta_0)$ and $u_t(\theta^t) = \widetilde{u}_{t-1}((\theta_1, ..., \theta_t); w^*(\theta_0), \varepsilon)$ for $t \geq 1$. Then:

$$\begin{aligned}
k(v) &\geq \sum_{t=0}^{\infty} \hat{\beta}^t E_{-1}\left[\theta_t u_t\left(\theta^t\right) - \eta C\left(u_t\left(\theta^t\right)\right)\right] \\
&\geq E_{-1}\left[\theta_0 u^*(\theta_0) - \eta C(u^*(\theta_0)) + \right. \\
&\quad \left. + \hat{\beta} \sum_{t=0}^{\infty} \hat{\beta}^t E_0\left[\theta_{t+1} u_{t+1}\left(\theta^{t+1}\right) \; - \eta C\left(u_{t+1}\left(\theta^{t+1}\right)\right)\right]\right] \\
&\geq \max_{u,w} E\left[\theta u(\theta) - \eta C(u(\theta)) + \hat{\beta} k(w(\theta)) - \hat{\beta}\varepsilon\right]
\end{aligned}$$

Since ε was arbitrary, it follows that:

$$k(v) \geq \max_{u,w} \quad E\left[\theta u(\theta) - \eta C(u(\theta)) + \hat{\beta} k(w(\theta))\right]$$

□