# CHAPTER 17

# EXCHANGE

In Chapter 13 we discussed the economic theory of a single market. We saw that when there were many economic agents each might reasonably be assumed to take market prices as outside of their control. Given these exogenous prices, each agent could then determine his or her demands and supplies for the good in question. The price adjusted to clear the market, and at such an equilibrium price, no agent would desire to change his or her actions.

The single-market story described above is a **partial equilibrium** model in that all prices other than the price of the good being studied are assumed to remain fixed. In the **general equilibrium** model *all* prices are variable, and equilibrium requires that *all* markets clear. Thus, general equilibrium theory takes account of all of the interactions between markets, as well as the functioning of the individual markets.

In the interests of exposition, we will examine first the special case of the general equilibrium model where all of the economic agents are consumers. This situation, known as the case of **pure exchange**, contains many of the phenomena present in the more extensive case involving firms and production.

In a pure exchange economy we have several consumers, each described by their preferences and the goods that they possess. The agents trade the goods among themselves according to certain rules and attempt to make themselves better off.

What will be the outcome of such a process? What are desirable outcomes of such a process? What allocative mechanisms are appropriate for achieving desirable outcomes? These questions involve a mixture of both positive and normative issues. It is precisely the interplay between the two types of questions that provides much of the interest in the theory of resource allocation.

## 17.1 Agents and goods

The concept of good considered here is very broad. Goods can be distinguished by time, location, and state of world. Services, such as labor services, are taken to be just another kind of good. There is assumed to be a market for each good, in which the price of that good is determined.

In the pure exchange model the only kind of economic agent is the consumer. Each consumer $i$ is described completely by his preference, $\succeq_i$ (or his utility function, $u_i$), and his **initial endowment** of the $k$ commodities, $\boldsymbol{\omega}_i$. Each consumer is assumed to behave competitively—that is, to take prices as given, independent of his or her actions. We assume that each consumer attempts to choose the most preferred bundle that he or she can afford.

The basic concern of the theory of general equilibrium is how goods are allocated among the economic agents. The amount of good $j$ that agent $i$ holds will be denoted by $x_i^j$. Agent $i$'s **consumption bundle** will be denoted by $\mathbf{x}_i = (x_i^1, \ldots, x_i^k)$; it is a $k$-vector describing how much of each good agent $i$ consumes. An **allocation** $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$ is a collection of $n$ consumption bundles describing what each of the $n$ agents holds. A **feasible allocation** is one that is physically possible; in the pure exchange case, this is simply an allocation that uses up all the goods, i.e., one in which $\sum_{i=1}^n \mathbf{x}_i = \sum_{i=1}^n \boldsymbol{\omega}_i$. (In some cases it is convenient to consider an allocation feasible if $\sum_{i=1}^n \mathbf{x}_i \leq \sum_{i=1}^n \boldsymbol{\omega}_i$.)

When there are two goods and two agents, we can use a convenient way of representing allocations, preferences, and endowments in a two-dimensional form, known as the **Edgeworth box**. We've depicted an example of an Edgeworth box in Figure 17.1.

Suppose that the total amount of good 1 is $\omega^1 = \omega_1^1 + \omega_2^1$ and that the total amount of good 2 is $\omega^2 = \omega_1^2 + \omega_2^2$. The Edgeworth box has a width of $\omega^1$ and a height of $\omega^2$. A point in the box, $(x_1^1, x_1^2)$, indicates how much agent 1 holds of the two goods. At the same time, it indicates the amount that agent 2 holds of the two goods: $(x_2^1, x_2^2) = (\omega^1 - x_1^1, \omega^2 - x_1^2)$. Geometrically, we measure agent 1's bundle from the lower left-hand corner

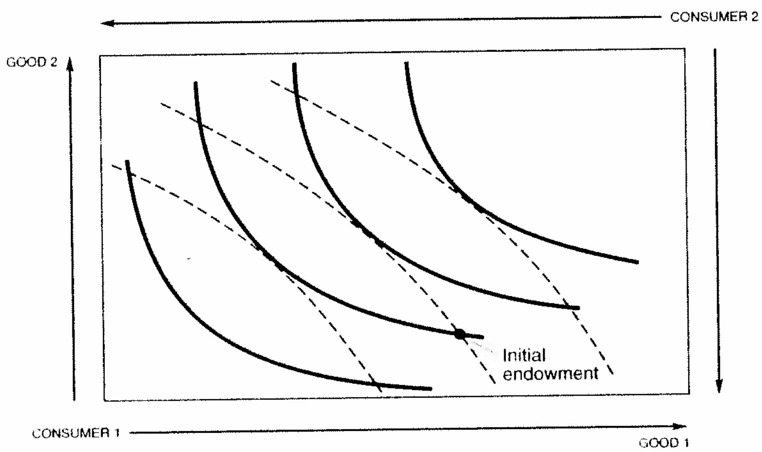



**Figure 17.1**

**Edgeworth box.** The length of the horizontal axis measures the total amount of good 1, and the height of the vertical axis measures the total amount of good 2. Each point in this box is a feasible allocation.

of the box. Agent 2's holdings are measured from the upper right-hand corner of the box. In this way, every feasible allocation of the two goods between the two agents can be represented by a point in this box.

We can also illustrate the agents' indifference curves in the box. There will be two sets of indifference curves, one set for each of the agents. All of the information contained in a two-person, two-good pure exchange economy can in this way be represented in a convenient graphical form.

## 17.2 Walrasian equilibrium

We have argued that, when there are many agents, it is reasonable to suppose that each agent takes the market prices as independent of his or her actions. Consider the particular case of pure exchange being described here. We imagine that there is some vector of market prices $\mathbf{p} = (p_1, \cdots, p_k)$, one price for each good. Each consumer takes these prices as given and chooses the most preferred bundle from his or her consumption set; that is, each consumer $i$ acts as if he or she were solving the following problem:

$$\max_{\mathbf{x}_i} u_i(\mathbf{x}_i)$$
$$\text{such that } \mathbf{p}\mathbf{x}_i = \mathbf{p}\boldsymbol{\omega}_i.$$

The answer to this problem, $\mathbf{x}_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}_i)$, is the consumer's **demand function**, which we have studied in Chapter 9. In that chapter the consumer's income or wealth, $m_i$ was exogenous. Here we take the consumer's wealth to be the market value of his or her initial endowment, so that $m_i = \mathbf{p}\boldsymbol{\omega}_i$. We saw in Chapter 9 that under an assumption of strict convexity of preferences, the demand functions will be well-behaved continuous functions.

Of course, for an arbitrary price vector $\mathbf{p}$, it may not be possible actually to make the desired transactions for the simple reason that the aggregate demand, $\sum_i \mathbf{x}_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}_i)$, may not be equal to the aggregate supply, $\sum_i \boldsymbol{\omega}_i$.

It is natural to think of an equilibrium price vector as being one that clears all markets; that is, a set of prices for which demand equals supply in every market. However, this is a bit too strong for our purposes. For example, consider the case where some of the goods are undesirable. In this case, they may well be in excess supply in equilibrium.

For this reason, we typically define a **Walrasian equilibrium** to be a pair $(\mathbf{p}^*, \mathbf{x}^*)$, such that

$$\sum_i \mathbf{x}_i(\mathbf{p}^*, \mathbf{p}^*\boldsymbol{\omega}_i) \leq \sum_i \boldsymbol{\omega}_i.$$

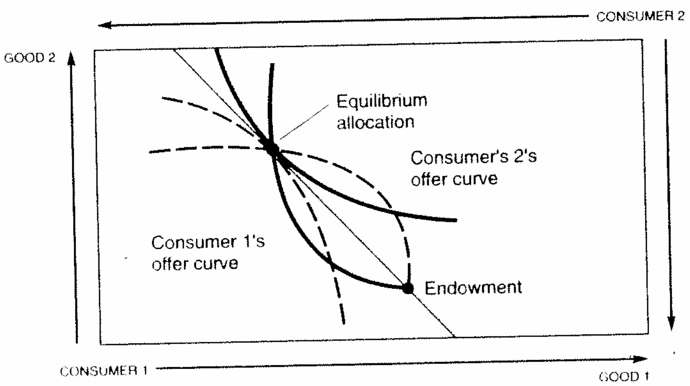
That is, $\mathbf{p}^*$ is a Walrasian equilibrium if there is no good for which there is positive excess demand. We show later, in Chapter 17, page 318, that if all goods are desirable—in a sense to be made precise—then in fact demand will equal supply in all markets.

## 17.3 Graphical analysis

Walrasian equilibria can be examined geometrically by use of the Edgeworth box. Given any price vector, we can determine the budget line of each agent and use the indifference curves to find the demanded bundles of each agent. We then search for a price vector such that the demanded points of the two agents are compatible.

In Figure 17.2 we have drawn such an equilibrium allocation. Each agent is maximizing his utility on his budget line and these demands are compatible with the total supplies available. Note that the Walrasian equilibrium occurs at a point where the two indifference curves are tangent. This is clear, since utility maximization requires that each agent's marginal rate of substitution be equal to the common price ratio.

Another way to describe equilibrium is through the use of **offer curves**. Recall that a consumer's offer curve describes the locus of tangencies between the indifference curves and the budget line as the relative prices vary—i.e., the set of demanded bundles. Thus, at an equilibrium in the Edgeworth box the offer curves of the two agents intersect. At such an intersection the demanded bundles of each agent are compatible with the available supplies.




**Walrasian equilibrium in the Edgeworth box.** Each agent is maximizing utility on his budget line.

**Figure 17.2**

## 17.4 Existence of Walrasian equilibria

Will there always exist a price vector where all markets clear? We will analyze this question of the **existence of Walrasian equilibria** in this section.

Let us recall a few facts about this existence problem. First of all, the budget set of a consumer remains unchanged if we multiply all prices by any positive constant; thus, each consumer's demand function has the property that $\mathbf{x}_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}_i) = \mathbf{x}_i(k\mathbf{p}, k\mathbf{p}\boldsymbol{\omega}_i)$ for all $k > 0$; i.e., the demand function is homogeneous of degree zero in prices. As the sum of homogeneous functions is homogeneous, the aggregate excess demand function,

$$\mathbf{z}(\mathbf{p}) = \sum_{i=1}^{n} [\mathbf{x}_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}_i) - \boldsymbol{\omega}_i],$$

is also homogeneous of degree zero in prices. Note that we ignore the fact that $\mathbf{z}$ depends on the vector of initial endowments, $(\boldsymbol{\omega}_i)$, since the initial endowments remain constant in the course of our analysis.

If all of the individual demand functions are continuous, then $\mathbf{z}$ will be a continuous function, since the sum of continuous functions is a continuous function. Furthermore, the aggregate excess demand function must satisfy a condition known as **Walras' law.**

**Walras' law.** *For any price vector* $\mathbf{p}$, *we have* $\mathbf{p}\mathbf{z}(\mathbf{p}) \equiv 0$; *i.e., the value of the excess demand is identically zero.*

*Proof.* We simply write the definition of aggregate excess demand and multiply by $\mathbf{p}$:

$$\mathbf{p}\mathbf{z}(\mathbf{p}) = \mathbf{p}\left[\sum_{i=1}^{n} \mathbf{x}_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}_i) - \sum_{i=1}^{n} \boldsymbol{\omega}_i\right] = \sum_{i=1}^{n} [\mathbf{p}\mathbf{x}_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}_i) - \mathbf{p}\boldsymbol{\omega}_i] = 0,$$

since $\mathbf{x}_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}_i)$ must satisfy the budget constraint $\mathbf{p}\mathbf{x}_i = \mathbf{p}\boldsymbol{\omega}_i$ for each agent $i = 1, \ldots, n$. ∎

Walras' law says something quite obvious: if each individual satisfies his budget constraint, so that the value of his excess demand is zero, then the value of the *sum* of the excess demands must be zero. It is important to realize that Walras' law asserts that the value of excess demand is *identically* zero—the value of excess demand is zero for *all* prices.

Combining Walras' law and the definition of equilibrium, we have two useful propositions.

**Market clearing.** *If demand equals supply in $k-1$ markets, and $p_k > 0$, then demand must equal supply in the $k^{th}$ market.*

*Proof.* If not, Walras' law would be violated. ∎

**Free goods.** *If $\mathbf{p}^*$ is a Walrasian equilibrium and $z_j(\mathbf{p}^*) < 0$, then $p_j^* = 0$. That is, if some good is in excess supply at a Walrasian equilibrium it must be a free good.*

*Proof.* Since $\mathbf{p}^*$ is a Walrasian equilibrium, $\mathbf{z}(\mathbf{p}^*) \leq \mathbf{0}$. Since prices are nonnegative, $\mathbf{p}^*\mathbf{z}(\mathbf{p}^*) = \sum_{i=1}^{k} p_i^* z_i(\mathbf{p}^*) \leq 0$. If $z_j(\mathbf{p}^*) < 0$ and $p_j^* > 0$, we would have $\mathbf{p}^*\mathbf{z}(\mathbf{p}^*) < 0$, contradicting Walras' law. ∎

This proposition shows us what conditions are required for all markets to clear in equilibrium. Suppose that all goods are desirable in the following sense:

**Desirability.** *If $p_i = 0$, then $z_i(\mathbf{p}) > 0$ for $i = 1, \ldots, k$.*

The desirability assumption says that if some price is zero, the aggregate excess demand for that good is strictly positive. Then we have the following proposition:

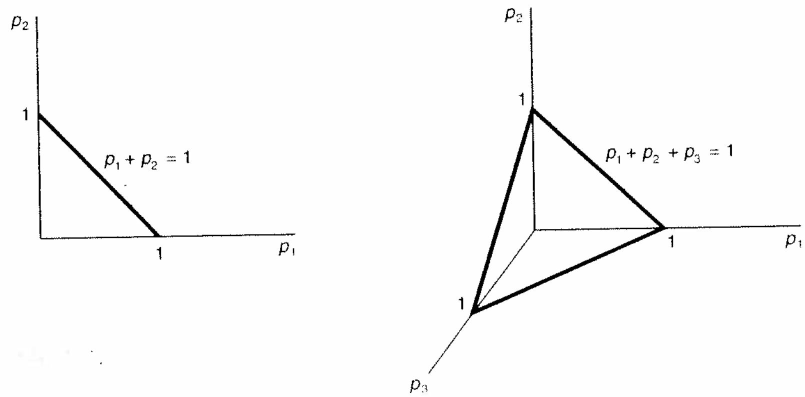



**Figure 17.3** **Price simplices.** The first panel depicts the one-dimensional price simplex $S^1$; the second panel depicts $S^2$.

**Equality of demand and supply.** *If all goods are desirable and* $\mathbf{p}^*$ *is a Walrasian equilibrium, then* $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

*Proof.* Assume $z_i(\mathbf{p}^*) < 0$. Then by the free goods proposition, $p_i^* = 0$. But then by the desirability assumption, $z_i(\mathbf{p}^*) > 0$, a contradiction. ∎

To summarize: in general all we require for equilibrium is that there is no excess demand for any good. But the above propositions indicate that if some good is actually in excess supply in equilibrium, then its price must be zero. Thus, if each good is desirable in the sense that a zero price implies it will be in excess demand, then equilibrium will in fact be characterized by the equality of demand and supply in every market.

## 17.5 Existence of an equilibrium

Since the aggregate excess demand function is homogeneous of degree zero, we can normalize prices and express demands in terms of **relative prices**. There are several ways to do this, but a convenient normalization for our purposes is to replace each absolute price $\hat{p}_i$ by a normalized price

$$p_i = \frac{\hat{p}_i}{\sum_{j=1}^{k} \hat{p}_j}.$$

This has the consequence that the normalized prices $p_i$ must always sum up to 1. Hence, we can restrict our attention to price vectors belonging to the $k-1$-dimensional unit simplex:

$$S^{k-1} = \left\{ \mathbf{p} \text{ in } R_+^k : \sum_{i=1}^{k} p_i = 1 \right\}.$$

For a picture of $S^1$ and $S^2$ see Figure 17.3.

We return now to the question of the existence of Walrasian equilibrium: is there a $\mathbf{p}^*$ that clears all markets? Our proof of existence makes use of the Brouwer fixed-point theorem.

**Brouwer fixed-point theorem.** *If* $\mathbf{f} : S^{k-1} \to S^{k-1}$ *is a continuous function from the unit simplex to itself, there is some* $\mathbf{x}$ *in* $S^{k-1}$ *such that* $\mathbf{x} = \mathbf{f}(\mathbf{x})$.

*Proof.* The proof for the general case is beyond the scope of this book; a good proof is in Scarf (1973). However, we will prove the theorem for $k = 2$.

In this case, we can identify the unit 1-dimensional simplex $S^1$ with the unit interval. According to the setup of the theorem we have a continuous function $f: [0,1] \to [0,1]$ and we want to establish that there is some $x$ in $[0,1]$ such that $x = f(x)$.

Consider the function $g$ defined by $g(x) = f(x) - x$. Geometrically, $g$ just measures the difference between $f(x)$ and the diagonal in the box depicted in Figure 17.4. A fixed point of the mapping $f$ is an $x^*$ where $g(x^*) = 0$.

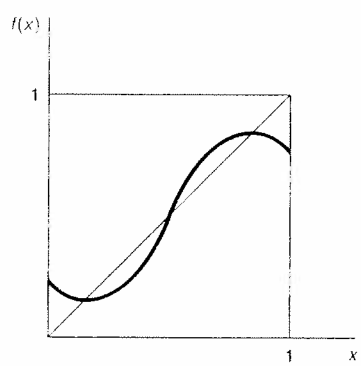



**Figure 17.4** **Proof of Brouwer's theorem in two dimensions.** In the case depicted, there are three points where $x = f(x)$.

Now $g(0) = f(0) - 0 \geq 0$ since $f(0)$ is in $[0,1]$, and $g(1) = f(1) - 1 \leq 0$ for the same reason. Since $f$ is continuous, we can apply the intermediate value theorem and conclude that there is some $x$ in $[0,1]$ such that $g(x) = f(x) - x = 0$, which proves the theorem. ∎

We are now in a position to prove the main existence theorem.

**Existence of Walrasian equilibria.** *If* $\mathbf{z} : S^{k-1} \to R^k$ *is a continuous function that satisfies Walras' law,* $\mathbf{pz}(\mathbf{p}) \equiv 0$*, then there exists some* $\mathbf{p}^*$ *in* $S^{k-1}$ *such that* $\mathbf{z}(\mathbf{p}^*) \leq 0$.

*Proof.* Define a map $g : S^{k-1} \to S^{k-1}$ by

$$g_i(\mathbf{p}) = \frac{p_i + \max(0, z_i(\mathbf{p}))}{1 + \sum_{j=1}^k \max(0, z_j(\mathbf{p}))} \quad \text{for } i = 1, \ldots, k.$$

Notice that this map is continuous since $z$ and the max function are continuous functions. Furthermore, $\mathbf{g}(\mathbf{p})$ is a point in the simplex $S^{k-1}$ since $\sum_i g_i(\mathbf{p}) = 1$. This map also has a reasonable economic interpretation: if there is excess demand in some market, so that $z_i(\mathbf{p}) \geq 0$, then the relative price of that good is increased.

By Brouwer's fixed-point theorem there is a $\mathbf{p}^*$ such that $\mathbf{p}^* = \mathbf{g}(\mathbf{p}^*)$; i.e.,

$$p_i^* = \frac{p_i^* + \max(0, z_i(\mathbf{p}^*))}{1 + \sum_j \max(0, z_j(\mathbf{p}^*))} \quad \text{for } i = 1, \ldots, k. \tag{17.1}$$

We will show that $\mathbf{p}^*$ is a Walrasian equilibrium. Cross-multiply equation (17.1) and rearrange to get

$$p_i^* \sum_{j=1}^k \max(0, z_j(\mathbf{p}^*)) = \max(0, z_i(\mathbf{p}^*)) \quad i = 1, \ldots, k.$$

Now multiply each of these $k$ equations by $z_i(\mathbf{p}^*)$:

$$z_i(\mathbf{p}^*) p_i^* \left[ \sum_{j=1}^k \max(0, z_j(\mathbf{p}^*)) \right] = z_i(\mathbf{p}^*) \max(0, z_i(\mathbf{p}^*)) \quad i = 1, \ldots, k.$$

Sum these $k$ equations to get

$$\left[ \sum_{j=1}^k \max(0, z_j(\mathbf{p}^*)) \right] \sum_{i=1}^k p_i^* z_i(\mathbf{p}^*) = \sum_{i=1}^k z_i(\mathbf{p}^*) \max(0, z_i(\mathbf{p}^*)).$$

Now $\sum_{i=1}^k p_i^* z_i(\mathbf{p}^*) = 0$ by Walras' law so we have

$$\sum_{i=1}^k z_i(\mathbf{p}^*) \max(0, z_i(\mathbf{p}^*)) = 0.$$

Each term of this sum is greater than or equal to zero since each term is either 0 or $(z_i(\mathbf{p}^*))^2$. But if any term were *strictly* greater than zero, the equality wouldn't hold. Hence, every term must be equal to zero, which says

$$z_i(\mathbf{p}^*) \leq 0 \quad \text{for } i = 1, \ldots, k.$$

∎

It is worth emphasizing the very general nature of the above theorem. All that is needed is that the excess demand function be continuous and satisfy Walras' law. Walras' law arises directly from the hypothesis that the consumer has to meet some kind of budget constraint; such behavior would seem to be necessary in any type of economic model. The hypothesis of continuity is more restrictive but not unreasonably so. We have seen earlier that if consumers all have strictly convex preferences then their demand functions will be well defined and continuous. The aggregate demand function will therefore be continuous. But even if the individual demand functions display discontinuities it may still turn out the aggregate demand function is continuous if there are a large number of consumers. Thus, continuity of aggregate demand seems like a relatively weak requirement.

However, there is one slight problem with the above argument for existence. It is true that aggregate demand is likely to be continuous for *positive* prices, but it is rather unreasonable to assume it is continuous even when some price goes to zero. If, for example, preferences were monotonic and the price of some good is zero, we would expect that the demand for such a good might be infinite. Thus, the excess demand function might not even be well defined on the boundary of the price simplex—i.e., on that set of price vectors where some prices are zero. However, this sort of discontinuity can be handled by using a slightly more complicated mathematical argument.

## EXAMPLE: The Cobb-Douglas Economy

Let agent 1 have utility function $u_1(x_1^1, x_1^2) = (x_1^1)^a (x_1^2)^{1-a}$ and endowment $\boldsymbol{\omega}_1 = (1, 0)$. Let agent 2 have utility function $u_2(x_2^1, x_2^2) = (x_2^1)^b (x_2^2)^{1-b}$ and endowment $\boldsymbol{\omega}_2 = (0, 1)$. Then agent 1's demand function for good 1 is

$$x_1^1(p_1, p_2, m_1) = \frac{am_1}{p_1}.$$

At prices $(p_1, p_2)$, income is $m_1 = p_1 \times 1 + p_2 \times 0 = p_1$. Substituting, we have

$$x_1^1(p_1, p_2) = \frac{ap_1}{p_1} = a.$$

Similarly, agent 2's demand function for good 1 is

$$x_2^1(p_1, p_2) = \frac{bp_2}{p_1}.$$

The equilibrium price is where total demand for each good equals total supply. By Walras' law, we only need find the price where total demand for good 1 equals total supply of good 1:

$$x_1^1(p_1, p_2) + x_2^1(p_1, p_2) = 1$$
$$a + \frac{bp_2}{p_1} = 1$$
$$\frac{p_2^*}{p_1^*} = \frac{1-a}{b}.$$

Note that, as usual, only relative prices are determined in equilibrium.

## 17.6 The first theorem of welfare economics

The existence of Walrasian equilibria is interesting as a positive result insofar as we believe the behavioral assumptions on which the model is based. However, even if this does not seem to be an especially plausible assumption in many circumstances, we may still be interested in Walrasian equilibria for their normative content. Let us consider the following definitions.

**Definitions of Pareto efficiency.** *A feasible allocation* $\mathbf{x}$ *is a* **weakly Pareto efficient** *allocation if there is no feasible allocation* $\mathbf{x}'$ *such that all agents strictly prefer* $\mathbf{x}'$ *to* $\mathbf{x}$. *A feasible allocation* $\mathbf{x}$ *is a* **strongly Pareto efficient** *allocation if there is no feasible allocation* $\mathbf{x}'$ *such that all agents weakly prefer* $\mathbf{x}'$ *to* $\mathbf{x}$, *and some agent strictly prefers* $\mathbf{x}'$ *to* $\mathbf{x}$.

It is easy to see that an allocation that is strongly Pareto efficient is also weakly Pareto efficient. In general, the reverse is not true. However, under some additional weak assumptions about preferences the reverse implication *is* true, so the concepts can be used interchangeably.

**Equivalence of weak and strong Pareto efficiency.** *Suppose that preferences are continuous and monotonic. Then an allocation is weakly Pareto efficient if and only if it is strongly Pareto efficient.*

*Proof.* If an allocation is strongly Pareto efficient, then it is certainly weakly Pareto efficient: if you can't make one person better off without hurting someone else, you certainly can't make everyone better off.

We need to show that if an allocation is weakly Pareto efficient, then it is strongly Pareto efficient. We prove the logically equivalent claim that if an allocation is *not* strongly efficient, then it is not weakly efficient.

Suppose, then, that it is possible to make some particular agent $i$ better off without hurting any other agents. We must demonstrate a way to make everyone better off. To do this, simply scale back $i$'s consumption bundle by a small amount and redistribute the goods taken from $i$ equally to the other agents. More precisely, replace $i$'s consumption bundle $\mathbf{x}_i$ by $\theta\mathbf{x}$, and replace each other agent $j$'s consumption bundle by $\mathbf{x}_j + (1-\theta)\mathbf{x}_i/(n-1)$. By continuity of preferences, it is possible to choose $\theta$ close enough to 1 so that agent $i$ is still better off. By monotonicity, all the other agents are made strictly better off by receiving the redistributed bundle. ∎

It turns out that the concept of weak Pareto efficiency is slightly more convenient mathematically, so we will generally use this definition: when we say "Pareto efficient" we generally mean "weakly Pareto efficient." However, we will henceforth always assume preferences are continuous and monotonic so that either definition is applicable.

Note that the concept of Pareto efficiency is quite weak as a normative concept; an allocation where one agent gets everything there is in the economy and all other agents get nothing will be Pareto efficient, assuming the agent who has everything is not satiated.

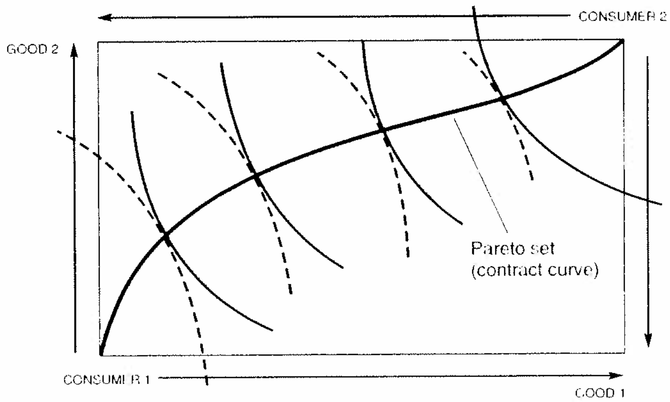
Pareto efficient allocations can easily be depicted in the Edgeworth box diagram introduced earlier. We only need note that, in the two-person case, Pareto efficient allocations can be found by fixing one agent's utility function at a given level and maximizing the other agent's utility function subject to this constraint. Formally, we only need solve the following maximization problem:

$$\max_{\mathbf{x}_1, \mathbf{x}_2} u_1(\mathbf{x}_1)$$
$$\text{such that } u_2(\mathbf{x}_2) \geq \overline{u}_2$$
$$\mathbf{x}_1 + \mathbf{x}_2 = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2.$$

This problem can be solved by inspection in the Edgeworth box case. Simply find the point on one agent's indifference curve where the other agent reaches the highest utility. By now it should be clear that the resulting Pareto efficient point will be characterized by a tangency condition: the marginal rates of substitution must be the same for each agent.

For each fixed value of agent 2's utility, we can find an allocation where agent 1's utility is maximized and thus the tangency condition will be satisfied. The set of Pareto efficient points—the **Pareto set**—will thus be the locus of tangencies drawn in the Edgeworth box depicted in Figure 17.5. The **Pareto set** is also known as the **contract curve**, since it gives the set of efficient "contracts" or allocations.

The comparison of Figure 17.5 with Figure 17.2 reveals a striking fact: there seems to be a one-to-one correspondence between the set of Walrasian equilibria and the set of Pareto efficient allocations. Each Walrasian equilibrium satisfies the first-order condition for utility maximization that the marginal rate of substitution between the two goods for each agent be equal to the price ratio between the two goods. Since all agents face the




**Pareto efficiency in the Edgeworth box.** The Pareto set, or the contract curve, is the set of all Pareto efficient allocations.

**Figure 17.5**

same price ratio at a Walrasian equilibrium, all agents must have the same marginal rates of substitution.

Furthermore, if we pick an arbitrary Pareto efficient allocation, we know that the marginal rates of substitution must be equal across the two agents, and we can thus pick a price ratio equal to this common value. Graphically, given a Pareto efficient point we simply draw the common tangency line separating the two indifference curves. We then pick any point on this tangent line to serve as an initial endowment. If the agents try to maximize preferences on their budget sets, they will end up precisely at the Pareto efficient allocation.

The next two theorems give this correspondence precisely. First, we restate the definition of a Walrasian equilibrium in a more convenient form:

**Definition of Walrasian equilibrium.** *An allocation-price pair* $(\mathbf{x}, \mathbf{p})$ *is a* **Walrasian equilibrium** *if (1) the allocation is feasible, and (2) each agent is making an optimal choice from his budget set. In equations:*

(1) $\sum_{i=1}^{n} \mathbf{x}_i = \sum_{i=1}^{n} \boldsymbol{\omega}_i$.

(2) *If* $\mathbf{x}'_i$ *is preferred by agent* $i$ *to* $\mathbf{x}_i$, *then* $\mathbf{p}\mathbf{x}'_i > \mathbf{p}\boldsymbol{\omega}_i$.

This definition is equivalent to the original definition of Walrasian equilibrium, as long as the desirability assumption is satisfied. This definition allows us to neglect the possibility of free goods, which are a bit of a nuisance for the arguments that follow.

**First Theorem of Welfare Economics.** *If* $(\mathbf{x}, \mathbf{p})$ *is a Walrasian equilibrium, then* $\mathbf{x}$ *is Pareto efficient.*

*Proof.* Suppose not, and let $\mathbf{x}'$ be a feasible allocation that all agents prefer to $\mathbf{x}$. Then by property 2 of the definition of Walrasian equilibrium, we have

$$\mathbf{p}\mathbf{x}_i' > \mathbf{p}\boldsymbol{\omega}_i \text{ for } i = 1, \ldots, n.$$

Summing over $i = 1, \ldots, n$, and using the fact that $\mathbf{x}'$ is feasible, we have

$$\mathbf{p}\sum_{i=1}^{n} \boldsymbol{\omega}_i = \mathbf{p}\sum_{i=1}^{n} \mathbf{x}_i' > \sum_{i=1}^{n} \mathbf{p}\boldsymbol{\omega}_i,$$

which is a contradiction. ∎

This theorem says that if the behavioral assumptions of our model are satisfied then the market equilibrium is efficient. A market equilibrium is not necessarily "optimal" in any ethical sense, since the market equilibrium may be very "unfair." The outcome depends entirely on the original distribution of endowments. What is needed is some further ethical criterion to choose among the efficient allocations. Such a concept, the concept of a welfare function, will be discussed later in this chapter.

## 17.7 The second welfare theorem

We have shown that every Walrasian equilibrium is Pareto efficient. Here we show that every Pareto efficient allocation is a Walrasian equilibrium.

**Second Theorem of Welfare Economics.** *Suppose* $\mathbf{x}^*$ *is a Pareto efficient allocation in which each agent holds a positive amount of each good. Suppose that preferences are convex, continuous, and monotonic. Then* $\mathbf{x}^*$ *is a Walrasian equilibrium for the initial endowments* $\boldsymbol{\omega}_i = \mathbf{x}_i^*$ *for* $i = 1, \ldots, n$.

*Proof.* Let

$$P_i = \{\mathbf{x}_i \text{ in } R^k : \mathbf{x}_i \succ_i \mathbf{x}_i^*\}.$$

This is the set of all consumption bundles that agent $i$ prefers to $\mathbf{x}_i^*$. Then define

$$P = \sum_{i=1}^{n} P_i = \left\{ \mathbf{z} : \mathbf{z} = \sum_{i=1}^{n} \mathbf{x}_i \text{ with } \mathbf{x}_i \text{ in } P_i \right\}.$$

$P$ is the set of all bundles of the $k$ goods that can be distributed among the $n$ agents so as to make each agent better off. Since each $P_i$ is a convex set by hypothesis and the sum of convex sets is convex, it follows that $P$ is a convex set.

Let $\boldsymbol{\omega} = \sum_{i=1}^{n} \mathbf{x}_i^*$ be the current *aggregate* bundle. Since $\mathbf{x}^*$ is Pareto efficient, there is no redistribution of $\mathbf{x}^*$ that makes everyone better off. This means that $\boldsymbol{\omega}$ is not an element of the set $P$.

Hence, by the separating hyperplane theorem (Chapter 26, page 483) there exists a $\mathbf{p} \neq \mathbf{0}$ such that

$$\mathbf{pz} \geq \mathbf{p} \sum_{i=1}^{n} \mathbf{x}_i^* \quad \text{for all } \mathbf{z} \text{ in } P.$$

Rearranging this equation gives us

$$\mathbf{p}\left(\mathbf{z} - \sum_{i=1}^{n} \mathbf{x}_i^*\right) \geq \mathbf{0} \quad \text{for all } \mathbf{z} \text{ in } P. \tag{17.2}$$

We want to show that $\mathbf{p}$ is in fact an equilibrium price vector. The proof proceeds in three steps.

(1) $\mathbf{p}$ is nonnegative; that is, $\mathbf{p} \geq \mathbf{0}$.

To see this, let $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)$ with a 1 in the $i^{th}$ component. Since preferences are monotonic, $\boldsymbol{\omega} + \mathbf{e}_i$ must lie in $P$; since if we have one more unit of any good, it is possible to redistribute it to make everyone better off. Inequality (17.2) then implies

$$\mathbf{p}(\boldsymbol{\omega} + \mathbf{e}_i - \boldsymbol{\omega}) \geq 0 \quad \text{for } i = 1, \ldots, k.$$

Canceling terms,

$$\mathbf{pe}_i \geq 0 \quad \text{for } i = 1, \ldots, k.$$

This equation implies $p_i \geq 0$ for $i = 1, \ldots, k$.

(2) If $\mathbf{y}_j \succ_j \mathbf{x}_j^*$, then $\mathbf{py}_j \geq \mathbf{px}_j^*$, for each agent $j = 1, \ldots, n$.

We already know that, if *every* agent $i$ prefers $\mathbf{y}_i$ to $\mathbf{x}_i^*$, then

$$\mathbf{p} \sum_{i=1}^{n} \mathbf{y}_i \geq \mathbf{p} \sum_{i=1}^{n} \mathbf{x}_i^*.$$

Now suppose only that some *particular* agent $j$ prefers some bundle $\mathbf{y}_j$ to $\mathbf{x}_j$. Construct an allocation $\mathbf{z}$ by taking some of each good away from agent $j$ and distributing it to the other agents. Formally, let $\theta$ be a small number, and define the allocations $\mathbf{z}$ by

$$z_j = (1 - \theta) y_j$$
$$z_i = x_i^* + \frac{\theta y_j}{n - 1} \quad i \neq j.$$

For small enough $\theta$, strong monotonicity implies the allocation $\mathbf{z}$ is Pareto preferred to $\mathbf{x}^*$, and thus $\sum_{i=1}^{n} \mathbf{z}_i$ lies in $P$. Applying inequality (17.2), we have

$$\mathbf{p}\sum_{i=1}^{n} \mathbf{z}_i \geq \mathbf{p}\sum_{i=1}^{n} \mathbf{x}_i^*$$

$$\mathbf{p}\left[\mathbf{y}_j(1-\theta) + \sum_{i \neq j} \mathbf{x}_i^* + \mathbf{y}_j\theta\right] \geq \mathbf{p}\left[\mathbf{x}_j^* + \sum_{i \neq j} \mathbf{x}_i^*\right]$$

$$\mathbf{p}\mathbf{y}_j \geq \mathbf{p}\mathbf{x}_j^*.$$

This argument demonstrates that if agent $j$ prefers $\mathbf{y}_j$ to $\mathbf{x}_j^*$, then $\mathbf{y}_j$ can cost no less than $\mathbf{x}_j^*$. It remains to show that we can make this inequality strict.

(3) If $\mathbf{y}_j \succ_j \mathbf{x}_j^*$, we must have $\mathbf{p}\mathbf{y}_j > \mathbf{p}\mathbf{x}_j^*$.

We already know that $\mathbf{p}\mathbf{y}_j \geq \mathbf{p}\mathbf{x}_j^*$; we want to rule out the possibility that the equality case holds. Accordingly, we will assume that $\mathbf{p}\mathbf{y}_j = \mathbf{p}\mathbf{x}_j^*$ and try to derive a contradiction.

From the assumption of continuity of preferences, we can find some $\theta$ with $0 < \theta < 1$ such that $\theta\mathbf{y}_j$ is strictly preferred to $\mathbf{x}_j^*$. By the argument of part (2), we know that $\theta\mathbf{y}_j$ must cost at least as much as $\mathbf{x}_j^*$:

$$\theta\mathbf{p}\mathbf{y}_j \geq \mathbf{p}\mathbf{x}_j^*. \tag{17.3}$$

One of the hypotheses of the theorem is that $\mathbf{x}_j^*$ has every component strictly positive; from this it follows that $\mathbf{p}\mathbf{x}_j^* > 0$.

Therefore, if $\mathbf{p}\mathbf{y}_j - \mathbf{p}\mathbf{x}_j^* = 0$, it follows that $\theta\mathbf{p}\mathbf{y}_j < \mathbf{p}\mathbf{x}_j^*$. But this contradicts (17.3), and concludes the proof of the theorem. ∎

It is worth considering the hypotheses of this proposition. Convexity and continuity of preferences are crucial, of course, but strong monotonicity can be relaxed considerably. One can also relax the assumption that $\mathbf{x}_i^* \gg \mathbf{0}$.

## A revealed preference argument

There is a very simple but somewhat indirect proof of the Second Welfare Theorem that is based on a revealed preference argument and the existence theorem described earlier in this chapter.

**Second Theorem of Welfare Economics.** *Suppose that* $\mathbf{x}^*$ *is a Pareto efficient allocation and that preferences are nonsatiated. Suppose further that a competitive equilibrium exists from the initial endowments* $\boldsymbol{\omega}_i = \mathbf{x}_i^*$ *and let it be given by* $(\mathbf{p}', \mathbf{x}')$. *Then, in fact,* $(\mathbf{p}', \mathbf{x}^*)$ *is a competitive equilibrium.*

*Proof.* Since $\mathbf{x}_i^*$ is in consumer $i$'s budget set by construction, we must have $\mathbf{x}_i' \succeq_i \mathbf{x}_i^*$. Since $\mathbf{x}^*$ is Pareto efficient, this implies that $\mathbf{x}_i^* \sim_i \mathbf{x}_i'$. Thus if $\mathbf{x}_i'$ is optimal, so is $\mathbf{x}_i^*$. Hence, $(\mathbf{p}', \mathbf{x}^*)$ is a Walrasian equilibrium. ∎

This argument shows that if a competitive equilibrium *exists* from a Pareto efficient allocation, then that Pareto efficient allocation is *itself* a competitive equilibrium. The remarks following the existence theorem in this chapter indicate that the only essential requirement for existence is continuity of the aggregate demand function. Continuity follows from either the convexity of individual preferences or the assumption of a "large" economy. Thus, the Second Welfare Theorem holds under the same circumstances.

## 17.8 Pareto efficiency and calculus

We have seen in the last section that every competitive equilibrium is Pareto efficient and essentially every Pareto efficient allocation is a competitive equilibrium for some distribution of endowments. In this section we will investigate this relationship more closely through the use of differential calculus. Essentially, we will derive first-order conditions that characterize market equilibria and Pareto efficiency and then compare these two sets of conditions.

The conditions characterizing the market equilibrium are very simple.

**Calculus characterization of equilibrium.** *If* $(\mathbf{x}^*, \mathbf{p}^*)$ *is a market equilibrium with each consumer holding a positive amount of every good, then there exists a set of numbers* $(\lambda_1, \ldots, \lambda_n)$ *such that:*

$$\mathbf{D}u_i(\mathbf{x}^*) = \lambda_i \mathbf{p}^* \qquad i = 1, \ldots, n.$$

*Proof.* If we have a market equilibrium, then each agent is maximized on his budget set, and these are just the first-order conditions for such utility maximization. The $\lambda_i$'s are the agents' marginal utilities of income. ∎

The first-order conditions for Pareto efficiency are a bit harder to formulate. However, the following trick is very useful.

**Calculus characterization of Pareto efficiency.** *A feasible allocation* $\mathbf{x}^*$ *is Pareto efficient if and only if* $\mathbf{x}^*$ *solves the following* $n$ *maximization problems for* $i = 1, \ldots, n$:

$$\max_{(x_i^g, x_j^g)} \; u_i(\mathbf{x}_i)$$

$$\text{such that } \sum_{h=1}^{n} x_h^g \leq \omega^g \quad g = 1, \ldots, k$$

$$u_j(\mathbf{x}_j^*) \leq u_j(\mathbf{x}_j) \quad j \neq i.$$

*Proof.* Suppose $\mathbf{x}^*$ solves all maximization problems but $\mathbf{x}^*$ is not Pareto efficient. This means that there is some allocation $\mathbf{x}'$ where everyone is better off. But then $\mathbf{x}^*$ couldn't solve any of the problems, a contradiction.

Conversely, suppose $\mathbf{x}^*$ is Pareto efficient, but it doesn't solve one of the problems. Instead, let $\mathbf{x}'$ solve that particular problem. Then $\mathbf{x}'$ makes one of the agents better off without hurting any of the other agents, which contradicts the assumption that $\mathbf{x}^*$ is Pareto efficient. ∎

Before examining the Lagrange formulation for one of these maximization problems, let's do a little counting. There are $k + n - 1$ constraints for each of the $n$ maximization problems. The first $k$ constraints are resource constraints, and the second $n - 1$ constraints are the utility constraints. In each maximization problem there are $kn$ choice variables: how much each of the $n$ agents has of each of the $k$ goods.

Let $q^g$, for $g = 1, \ldots, k$, be the Kuhn-Tucker multipliers for the resource constraints, and let $a_j$, for $j \neq i$, be the multipliers for the utility constraints. Write the Lagrangian for one of the maximization problems.

$$\mathcal{L} = u_i(\mathbf{x}_i) - \sum_{g=1}^{k} q^g \left[\sum_{i=1}^{n} x_i^g - \omega^g\right] - \sum_{j \neq i} a_j [u_j(\mathbf{x}_j^*) - u_j(\mathbf{x}_j)].$$

Now differentiate $\mathcal{L}$ with respect to $x_j^g$ where $g = 1, \ldots, k$ and $j = 1, \ldots, n$. We get first-order conditions of the form

$$\frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_i^g} - q^g = 0 \qquad g = 1, \ldots, k$$

$$a_j \frac{\partial u_j(\mathbf{x}_j^*)}{\partial x_j^g} - q^g = 0 \qquad j \neq i; g = 1, \ldots, k.$$

At first these conditions seem somewhat strange since they seem to be asymmetric. For each choice of $i$, we get different values for the multipliers $(q^g)$ and $(a_j)$. However, the paradox is resolved when we note that the *relative* values of the $q$s are independent of the choice of $i$. This is clear since the above conditions imply

$$\frac{\frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_i^g}}{\frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_i^h}} = \frac{q^g}{q^h} \qquad \text{for } i = 1, \ldots, n \text{ and } g, h = 1, \ldots, k.$$

Since $\mathbf{x}^*$ is given, $q^g/q^h$ must be independent of which maximization problem we solve. The same reasoning shows that $a_i/a_j$ is independent of which maximization problem we solve. The solution to the asymmetry problem now becomes clear: if we maximize agent $i$'s utility and use the other agent's utilities as constraints, then it is just as if we are arbitrarily setting agent $i$'s Kuhn-Tucker multiplier to be $a_i = 1$.

Using the First Welfare Theorem, we can derive nice interpretations of the weights $(a_i)$ and $(q^g)$: if $\mathbf{x}^*$ is a market equilibrium, then

$$\mathbf{D}u_i(\mathbf{x}_i^*) = \lambda_i \mathbf{p}^* \quad i = 1, \ldots, n.$$

However, all market equilibria are Pareto efficient and thus must satisfy

$$a_i \mathbf{D}u_i(\mathbf{x}_i^*) = \mathbf{q} \quad i = 1, \ldots, n.$$

From this it is clear that we can choose $\mathbf{p}^* = \mathbf{q}$ and $a_i = 1/\lambda_i$. In words, the Kuhn-Tucker multipliers on the resource constraints are just the competitive prices, and the Kuhn-Tucker multipliers on the agent's utilities are just the reciprocals of their marginal utilities of income.

If we eliminate the Kuhn-Tucker multipliers in the first-order conditions, we get the following conditions characterizing efficient allocations:

$$\frac{\frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_i^g}}{\frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_i^h}} = \frac{p_g^*}{p_h^*} = \frac{q^g}{q^h} \qquad i = 1, \ldots, n \text{ and } g, h = 1, \ldots, k.$$

This says that each Pareto efficient allocation must satisfy the condition that the marginal rate of substitution between each pair of goods is the same for every agent. This marginal rate of substitution is simply the ratio of the competitive prices.

The intuition behind this condition is fairly clear: if two agents had different marginal rates of substitution between some pair of goods, they could arrange a small trade that would make them both better off, contradicting the assumption of Pareto efficiency.

It is often useful to note that the first-order conditions for a Pareto efficient allocation are the same as the first-order conditions for maximizing a weighted sum of utilities. To see this, consider the problem

$$\max \sum_{i=1}^{n} a_i u_i(\mathbf{x}_i)$$

$$\text{such that } \sum_{i=1}^{n} x_i^g \leq \omega^g \quad g = 1, \ldots, k.$$

The first-order conditions for a solution to this problem are

$$a_i \mathbf{D} u_i(\mathbf{x}_i^*) = \mathbf{q}, \tag{17.4}$$

which are precisely the same as the necessary conditions for Pareto efficiency.

As the set of "welfare weights" $(a_1, \ldots, a_n)$ varies, we trace out the set of Pareto efficient allocations. If we are interested in conditions that characterize all Pareto efficient allocations, we need to manipulate the equations so that the welfare weights disappear. Generally, this boils down to expressing the conditions in terms of marginal rates of substitution.

Another way to see this is to think of incorporating the welfare weights into the definition of the utility function. If the original utility function for agent $i$ is $u_i(\mathbf{x}_i)$, take a monotonic transformation so that the new utility function is $v_i(\mathbf{x}_i) = a_i u_i(\mathbf{x}_i)$. The resulting first-order conditions characterize a *particular* Pareto efficient allocation—the one that maximizes the sum of utilities for a particular representation of utility. But if we manipulate the first-order conditions so that they are expressed in terms of marginal rates of substitution, we will typically find a condition that characterizes all efficient allocations.

For now we note that this calculus characterization of Pareto efficiency gives us a simple proof of the Second Welfare Theorem. Let us assume that all consumers have concave utility functions, although this is not really required. Then if $\mathbf{x}^*$ is a Pareto efficient allocation, we know from the first-order conditions that

$$\mathbf{D} u_i(\mathbf{x}^*) = \frac{1}{a_i} \mathbf{q} \text{ for } i = 1, \ldots, n.$$

Thus, the gradient of each consumer's utility function is proportional to some fixed vector $\mathbf{q}$. Let us choose $\mathbf{q}$ to be the vector of competitive prices. We need to check that each consumer is maximized on his budget set $\{\mathbf{x}_i : \mathbf{q}\mathbf{x}_i \leq \mathbf{q}\mathbf{x}_i^*\}$. But this follows quickly from concavity; according to the mathematical properties of concave functions:

$$u(\mathbf{x}_i) \leq u(\mathbf{x}_i^*) + \mathbf{D} u(\mathbf{x}_i^*)(\mathbf{x}_i - \mathbf{x}_i^*),$$

so

$$u(\mathbf{x}_i) \leq u(\mathbf{x}_i^*) + \frac{1}{a_i} \mathbf{q}(\mathbf{x}_i - \mathbf{x}_i^*).$$

Thus, if $\mathbf{x}_i$ is in the consumer's budget set, $u(\mathbf{x}) \leq u(\mathbf{x}_i^*)$.

## 17.9 Welfare maximization

One problem with the concept of Pareto efficiency as a normative criterion is that it is not very specific. Pareto efficiency is only concerned with efficiency and has nothing to say about distribution of welfare. Even if we agree that we should be at a Pareto efficient allocation, we still don't know which one we should be at.

One way to resolve these problems is to hypothesize the existence of some **social welfare function**. This is supposed to be a function that aggregates the individual utility functions to come up with a "social utility." The most reasonable interpretation of such a function is that it represents a social decision maker's preferences about how to trade off the utilities of different individuals. We will refrain from making philosophical comments here and just postulate that some such function exists; that is, we will suppose that we have

$$W : R^n \rightarrow R,$$

so that $W(u_1, \ldots, u_n)$ gives us the "social utility" resulting from any distribution $(u_1, \ldots, u_n)$ of private utilities. To make sense of this construction we have to pick a particular representation of each agent's utility which will be held fixed during the course of the discussion.

We will suppose that $W$ is increasing in each of its arguments—if you increase any agent's utility without decreasing anybody else's welfare, social welfare should increase. We suppose that society should operate at a point that maximizes social welfare; that is, we should choose an allocation $\mathbf{x}^*$ such that $\mathbf{x}^*$ solves

$$\max W(u_1(\mathbf{x}_1), \ldots, u_n(\mathbf{x}_n))$$
$$\text{such that } \sum_{i=1}^{n} x_i^g \leq \omega^g \quad g = 1, \ldots, k.$$

How do the allocations that maximize this welfare function compare to Pareto efficient allocations? The following is a trivial consequence of the monotonicity hypothesis:

**Welfare maximization and Pareto efficiency.** *If $\mathbf{x}^*$ maximizes a social welfare function, then $\mathbf{x}^*$ is Pareto efficient.*

*Proof.* If $\mathbf{x}^*$ were not Pareto efficient, then there would be some feasible allocation $\mathbf{x}'$ such that $u_i(\mathbf{x}_i') > u_i(\mathbf{x}_i^*)$ for $i = 1, \ldots, n$. But then $W(u_1(\mathbf{x}_1'), \ldots, u_n(\mathbf{x}_n')) > W(u_1(\mathbf{x}_1^*), \ldots, u_n(\mathbf{x}_n^*))$. ∎

Since welfare maxima are Pareto efficient, they must satisfy the same first-order conditions as Pareto efficient allocations; furthermore, under convexity assumptions, every Pareto efficient allocation is a competitive equilibrium, so the same goes for welfare maxima: every welfare maximum is a competitive equilibrium for some distribution of endowments.

This last observation gives us one further interpretation of the competitive prices: they are also the Kuhn-Tucker multipliers for the welfare maximization problem. Applying the envelope theorem, we see that the competitive prices measure the (marginal) social value of a good: how much welfare would increase if we had a small additional amount of the good. However, this is true only for the choice of welfare function that is maximized at the allocation in question.

We have seen above that every welfare maximum is Pareto efficient, but is the converse necessarily true? We saw in the last section that every Pareto efficient allocation satisfied the same first-order conditions as the problem of maximizing a weighted sum of utilities, so it might seem plausible that under convexity and concavity assumptions things might work out nicely. Indeed they do.

**Pareto efficiency and welfare maximization.** *Let $\mathbf{x}^*$ be a Pareto efficient allocation with $\mathbf{x}_i^* \gg \mathbf{0}$ for $i = 1, \ldots, n$. Let the utility functions $u_i$ be concave, continuous, and monotonic functions. Then there is some choice of weights $a_i^*$ such that $\mathbf{x}^*$ maximizes $\sum a_i^* u_i(\mathbf{x}_i)$ subject to the resource constraints. Furthermore, the weights are such that $a_i^* = 1/\lambda_i^*$ where $\lambda_i^*$ is the $i^{th}$ agent's marginal utility of income; that is, if $m_i$ is the value of agent i's endowment at the equilibrium prices $\mathbf{p}^*$, then*

$$\lambda_i^* = \frac{\partial v_i(\mathbf{p}^*, m_i)}{\partial m_i}.$$

*Proof.* Since $\mathbf{x}^*$ is Pareto efficient, it is a Walrasian equilibrium. There therefore exist prices $\mathbf{p}$ such that each agent is maximized on his or her budget set; this in turn implies

$$\mathbf{D}u_i(\mathbf{x}_i^*) = \lambda_i \mathbf{p}^* \quad \text{for } i = 1, \ldots, n.$$

Consider now the welfare maximization problem

$$\begin{aligned} &\max \sum_{i=1}^n a_i u_i(\mathbf{x}_i) \\ &\text{such that } \sum_{i=1}^n x_i^1 \le \sum_{i=1}^n x_i^{1*} \\ &\qquad\qquad \vdots \\ &\qquad\qquad \sum_{i=1}^n x_i^k \le \sum_{i=1}^n x_i^{k*}. \end{aligned}$$

According to the sufficiency theorem for concave constrained maximization problems (Chapter 27, page 504), $\mathbf{x}^*$ solves this problem if there exist nonnegative numbers $(q_1, \ldots, q_k) = \mathbf{q}$ such that

$$a_i \mathbf{D} u_i(\mathbf{x}_i^*) = \mathbf{q}.$$

If we choose $a_i = 1/\lambda_i$, then the prices $\mathbf{p}$ serve as the appropriate nonnegative numbers and the proof is done. ∎

The interpretation of the weights as reciprocals of the marginal utilities of income makes good economic sense. If some agent has a large income at some Pareto efficient allocation, then his marginal utility of income will be small and his weight in the implicit social welfare function will be large.

The above two propositions complete the set of relationships between market equilibria, Pareto efficient allocations, and welfare maxima. To recapitulate briefly:

(1) competitive equilibria are always Pareto efficient;

(2) Pareto efficient allocations are competitive equilibria under convexity assumptions and endowment redistribution;

(3) welfare maxima are always Pareto efficient;

(4) Pareto efficient allocations are welfare maxima under concavity assumptions for some choice of welfare weights.

Inspecting the above relationships we can see the basic moral: a competitive market system will give efficient allocations but this says nothing about distribution. The choice of distribution of income is the same as the choice of a reallocation of endowments, and this in turn is equivalent to choosing a particular welfare function.

## Notes

The general equilibrium model was first formulated by Walras (1954). The first proof of existence was due to Wald (1951); more general treatments of existence were provided by McKenzie (1954) and Arrow & Debreu (1954). The definitive modern treatments are Debreu (1959) and Arrow & Hahn (1971). The latter work contains numerous historical notes.

The basic welfare results have a long history. The proof of the first welfare theorem used here follows Koopmans (1957). The importance of convexity in the Second Theorem was recognized by Arrow (1951) and Debreu (1953). The differentiable treatment of efficiency was first developed rigorously by

Samuelson (1947). The relationship between welfare maxima and Pareto efficiency follows Negisihi (1960).

The revealed preference proof of the Second Welfare Theorem is due to Maskin & Roberts (1980).

## Exercises

17.1. Consider the revealed preference argument for the Second Welfare Theorem. Show that if preferences are strictly convex, then $\mathbf{x}_i' = \mathbf{x}_i^*$ for all $i = 1, \ldots, n$.

17.2. Draw an Edgeworth box example with an infinite number of prices that are Walrasian equilibria.

17.3. Consider Figure 17.6. Here $\mathbf{x}^*$ is a Pareto efficient allocation, but $\mathbf{x}^*$ cannot be supported by competitive prices. Which assumption of the Second Welfare Theorem is violated?

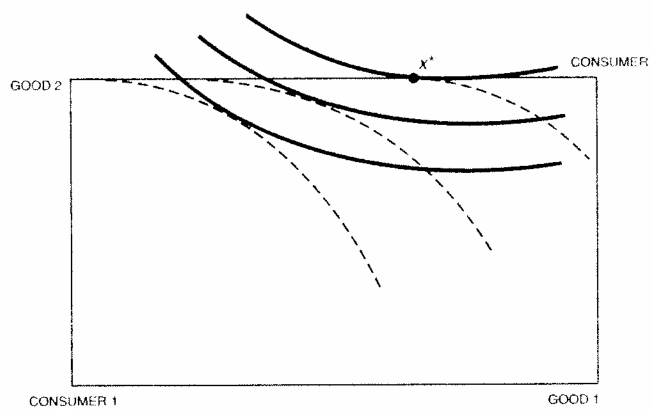



**Figure 17.6** **Arrow's exceptional case.** The allocation $\mathbf{x}^*$ is Pareto efficient but there are no prices at which $x^*$ is a Walrasian equilibrium.

17.4. There are two consumers $A$ and $B$ with the following utility functions and endowments:

$$u_A(x_A^1, x_A^2) = a \ln x_A^1 + (1-a) \ln x_A^2 \quad \boldsymbol{\omega}_A = (0, 1)$$
$$u_B(x_B^1, x_B^2) = \min(x_B^1, x_B^2) \quad \boldsymbol{\omega}_B = (1, 0).$$

Calculate the market clearing prices and the equilibrium allocation.

17.5. We have $n$ agents with identical strictly concave utility functions. There is some initial bundle of goods $\boldsymbol{\omega}$. Show that equal division is a Pareto efficient allocation.

17.6. We have two agents with *indirect* utility functions:

$$v_1(p_1, p_2, y) = \ln y - a \ln p_1 - (1 - a) \ln p_2$$
$$v_2(p_1, p_2, y) = \ln y - b \ln p_1 - (1 - b) \ln p_2$$

and initial endowments

$$\boldsymbol{\omega}_1 = (1, 1) \qquad \boldsymbol{\omega}_2 = (1, 1).$$

Calculate the market clearing prices.

17.7. Suppose that all consumers have quasilinear utility functions, so that $v_i(\mathbf{p}, m_i) = \nu_i(\mathbf{p}) + m_i$. Let $\mathbf{p}^*$ be a Walrasian equilibrium. Show that the aggregate demand curve for each good must be downward sloping at $\mathbf{p}^*$. More generally, show that the gross substitutes matrix must be negative semidefinite.

17.8. Suppose we have two consumers $A$ and $B$ with identical utility functions $u_A(x_1, x_2) = u_B(x_1, x_2) = \max(x_1, x_2)$. There are 1 unit of good 1 and 2 units of good 2. Draw an Edgeworth box that illustrates the strongly Pareto efficient and the (weakly) Pareto efficient sets.

17.9. Consider an economy with 15 consumers and 2 goods. Consumer 3 has a Cobb-Douglas utility function $u_3(x_3^1, x_3^2) = \ln x_3^1 + \ln x_3^2$. At a certain Pareto efficient allocation $x^*$, consumer 3 holds (10,5). What are the competitive prices that support the allocation $\mathbf{x}^*$?

17.10. If we allow for the possibility of satiation, the consumer's budget constraint takes the form $\mathbf{p}\mathbf{x}_i \leq \mathbf{p}\boldsymbol{\omega}_i$. Walras' law then becomes $\mathbf{p}\mathbf{z}(\mathbf{p}) \leq 0$ for all $\mathbf{p} \geq 0$. Show that the proof of existence of a Walrasian equilibrium given in the text still applies for this generalized form of Walras' law.

17.11. Person A has a utility function of $u_A(x_1, x_2) = x_1 + x_2$ and person B has a utility function $u_B(x_1, x_2) = \max(x_1, x_2)$. Agent A and agent B have identical endowments of $(1/2, 1/2)$.

(a) Illustrate this situation in an Edgeworth box diagram.

(b) What is the equilibrium relationship between $p_1$ and $p_2$?

(c) What is the equilibrium allocation?