Economics 4113, Spring 2009. Instructor: David Rahman, University of Minnesota.

Homework 1—Due February 9, 2009

1. Sketch the cone generated by the columns of the following matrix:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

What is the cone generated by the first and third columns of the matrix? If $b = (1, 0)$, decide whether or not the system $Ax = b$ has a solution $x \geq \mathbf{0}$.

2. Sketch the cone generated by the columns of the following matrix:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 3 & -2 \end{bmatrix}$$

3. Sketch the hyperplane generated by the following equation in $\mathbb{R}^2$:

$$x_1 - 2x_2 = 3.$$

Identify the set of vectors in $\mathbb{R}^2$ such that $x_1 - 2x_2 < 3$, as well as the set of points such that $x_1 - 2x_2 > 3$. Now repeat this exercise with the equation $x_1 + 2x_2 = 3$ instead.

4. Prove the following statements given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$:

   (a) Either $Ax \geq b$ has a non-negative solution x or there is a non-negative solution y such that $yA \leq 0$ and $yb > 0$, but not both.

   (b) Either $Ax = \mathbf{0}$, $\sum_i x_i = 1$ has a non-negative solution or there exists $y \in \mathbb{R}^m$ such that $yA \gg \mathbf{0}$, but not both.

   (c) Either $Ax = \mathbf{0}$, has a nonzero, non-negative solution or there exists $y \in \mathbb{R}^m$ such that $yA \gg \mathbf{0}$, but not both.

   (d) Either $Ax \leq b$ has a solution or there exists $y \in \mathbb{R}^m_+$ such that $yA \geq \mathbf{0}$ and $yb < 0$, but not both.

   (e) Either there exists $x \gg \mathbf{0}$ such that $Ax = \mathbf{0}$ or there exists $y \in \mathbb{R}^m$ such that $yA > \mathbf{0}$, but not both.

   (f) The system $Ax \ll b$ has a solution if and only if $y = \mathbf{0}$ is the only solution to $\{yA = \mathbf{0}, yb \leq 0, y \geq \mathbf{0}\}$.

   (g) Let $F = \{x \in \mathbb{R}^n : Ax \leq \mathbf{0}\}$, $c \in \mathbb{R}^n$ and $G = \{x \in \mathbb{R}^n : cx \leq 0\}$. Prove that $F \subset G$ if and only if there exists $y \in \mathbb{R}^m_+$ such that $c = yA$.

5. A *put option* gives the holder the right to sell a stock at a prearranged price, K. If tomorrow's stock price S satisfies $S < K$ then the option holder's optimal strategy is to exercise her put option, i.e., sell the stock at price K to obtain a benefit of $K - S$. Otherwise, if $S \geq K$ then the put option becomes worthless.

   (a) Sketch the graph of the terminal value of a put option with strike price K as a function of the terminal value of the stock S.

   (b) If the stock price increases by a factor of u with probability p and decreases by a factor of d with probability $1-p$ and interest rates on bonds are equal to r, determine the price of a put option.

6. Consider the following multi-period version of the previous model. There are three periods, indexed by $t = 0, 1, 2$, with $t = 0$ signifying today. There are two assets: a bond that pays an interest rate equal to r every period, and a stock. In every period, the price of the stock can either increase by a factor of u with probability p or decrease by a factor of d with probability $1 - p$.

   (a) A *European call* with maturity date T and strike price K gives its holder the right to buy a stock at price K on date T. Suppose that $T = 2$. Calculate the price of the European call ...

      i. ... in period $t = 1$ assuming that the stock price went up.

      ii. ... in period $t = 1$ assuming that the stock price went down.

      iii. ... in period $t = 0$.

   (b) An *American call* with maturity date T and strike price K gives its holder the right to buy a stock at price K on any date up to and including T. Suppose that $T = 2$. Calculate the price of the American call ...

      i. ... in period $t = 1$ assuming that the stock price went up.

      ii. ... in period $t = 1$ assuming that the stock price went down.

      iii. ... in period $t = 0$.

   (c) Is it ever optimal to exercise the American call early, i.e., at any period t less than T?

Acknowledgement: Questions 1,2 and parts of 4 were taken from Vohra's (2005) book, specifically his exercises at the end of Chapter 2.