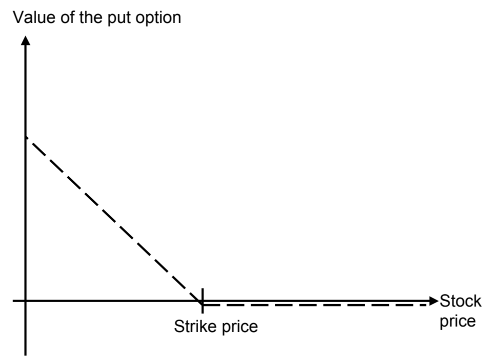
Economics 4113, Spring 2009. Instructor: David Rahman, University of Minnesota.

Homework 1—Suggested Answers to Questions 5 and 6

5 (a). The payoff from a put option with strike price K as a function of the stock price S is given by $\max\{K - S, 0\}$. Its graph looks like:



5 (b). Given the matrix of state-contingent payoffs

$$D = \begin{bmatrix} dS & uS \\ rB & rB \end{bmatrix}$$

and price vector $p = (S, B)$, to find a state-price vector when there is no arbitrage, we obtain the following two equations in two unknowns:

$$dS\pi_d + uS\pi_u = S \qquad rB\pi_d + rB\pi_u = B.$$

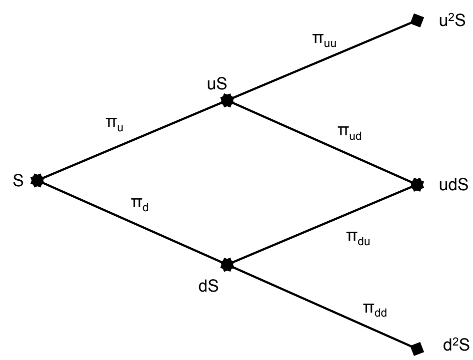
Solving the system, it follows that

$$\pi_d = \frac{u-r}{r(u-d)} \qquad \pi_u = \frac{r-d}{r(u-d)}.$$

Therefore, there is no arbitrage if and only if $u > r > d$. We assume that this holds throughout. In the absence of arbitrage, the price of a put option equals

$$\frac{u-r}{r(u-d)} \max\{K - dS, 0\} + \frac{r-d}{r(u-d)} \max\{K - uS, 0\}.$$

6. Consider the following two-period binomial tree:



Let C denote the price of a European call and C' the price of an American call.

(a) Let C_u be the price of a European call assuming that the stock went up in period 1. With exactly the same calculation as in the previous exercise, $\pi_{ud} = (u-r)/[r(u-d)]$ and $\pi_{uu} = (r-d)/[r(u-d)]$. Therefore,

$$C_u = \frac{u-r}{r(u-d)} \max\{udS - K, 0\} + \frac{r-d}{r(u-d)} \max\{u^2 S - K, 0\}.$$

Similarly, the price of the call if the stock went down in period 1 is

$$C_d = \frac{u-r}{r(u-d)} \max\{d^2 S - K, 0\} + \frac{r-d}{r(u-d)} \max\{udS - K, 0\}.$$

The same logic as that in Question 5 above implies that the following state prices apply from period 0 to period 1: $\pi_d = (u-r)/[r(u-d)]$ and $\pi_u = (r-d)/[r(u-d)]$. Therefore, the period-0 option price equals

$$C = \pi_d C_d + \pi_u C_u.$$

(b) An American option can be exercises anytime until maturity, therefore

$$C'_u = \max\{(uS - K)/r, C_u\} \qquad \text{and} \qquad C'_d = \max\{(dS - K)/r, C_d\},$$

since in each period the option holder can either exercise or hold the option. The option price in period 0 is finally given by

$$C' = \max\{(S-K)/r^2, \pi_d C'_d + \pi_u C'_u\},$$

where π_d and π_u are the state-price vectors derived in the previous part of the question.

(c) To find out whether or not it is optimal to ever exercise an American call early, let us begin in period 1, assuming that the stock price went up. Let $C_{uu} = \max\{u^2S - K, 0\}$ and $C_{ud} = \max\{udS - K, 0\}$. In this case,

$$\pi_u C_{uu} + \pi_d C_{ud} \geq \pi_u(u^2S - K) + \pi_d(udS - K) = uS - (K/r) \geq uS - K.$$

The left-most term equals C_u, the no-arbitrage price of a European call after the stock went up. The first inequality follows because $C_{uu} \geq u^2S - K$ and $C_{ud} \geq udS - K$. The subsequent equality follows because by absence of arbitrage, $\pi_u + \pi_d = 1/r$ and $u\pi_u + d\pi_d = 1$. Hence, $uS = \pi_u u^2 S + \pi_d udS$ and $K/r = \pi_u K + \pi_d K$. Finally, the last inequality follows because $r \geq 1$, so $K/r \leq K$. But the right-most term above represents the payoff from exercising the American call option early. Therefore, delaying exercise of the option is not optimal in period 1 after the stock went up. However, a similar argument applies in period 1 after the stock went down, as well as in the initial period.