Homework 2—Suggested Answers

4.2 There are several ways to write the dual. One way is the following:

$$\begin{aligned}
\min_{y_1,y_2,y_3} \quad 4y_1 + 2y_2 + 3y_3 \quad &\text{s.t.} \\
y_1 + y_2 + 2y_3 &\geq 1, \\
\tfrac{8}{3}y_1 + y_2 &\geq 2, \\
y_1 \geq 0, y_3 &\leq 0.
\end{aligned}$$

4.3 Given the primal problem

$$\begin{aligned}
\min_{x_1,x_2,x_3} \quad x_1 + x_2 - 3x_3 \quad &\text{s.t.} \\
x_1 + 2x_2 - 3x_3 &= 4, \\
4x_1 + 5x_2 - 9x_3 &= 13, \\
x_1, x_2, x_3 &\geq 0,
\end{aligned}$$

its dual is given by

$$\begin{aligned}
\max_{y_1,y_2} \quad 4y_1 + 13y_2 \quad &\text{s.t.} \\
y_1 + 4y_2 &\leq 1, \\
2y_1 + 5y_2 &\leq 1, \\
3y_1 + 9y_2 &\leq 3.
\end{aligned}$$

Since the first and second dual constraints together imply the third, eliminating the third constraint from the dual is without loss of generality. Adding the two times the first constraint to the third constraint, it follows that the value of the dual is at most 3. Solving the system

$$\begin{aligned}
y_1 + 4y_2 &= 1, \\
2y_1 + 5y_2 &= 1,
\end{aligned}$$

yields $(y_1, y_2) = (-\frac{1}{3}, \frac{1}{3})$, whose value equals 3. Therefore, it is an optimal dual solution. Notice that the third dual constraint is slack, so by complementary slackness the optimal primal solution has $x_3 = 0$. To solve the primal, we are therefore left with the following two equations in two unknowns:

$$\begin{aligned}
x_1 + 2x_2 &= 4, \\
4x_1 + 5x_2 &= 13.
\end{aligned}$$

Solving this system yields $x_1 = 2$ and $x_2 = 1$, which also gives a value of 3. Therefore, $(x_1, x_2, x_3) = (2, 1, 0)$ is an optimal primal solution.

4.4 Given the primal problem

$$\begin{aligned} \min_{x_1,x_2,x_3,x_4} \quad -x_1 + 2x_2 + 8x_3 + 2x_4 &\ \text{ s.t.} \\ -x_2 + x_3 + x_4 &\geq 1, \\ x_1 + 2x_2 - 2x_3 + x_4 &\leq 2, \\ x_1, x_2, x_3, x_4 &\geq 0, \end{aligned}$$

its dual is given by

$$\begin{aligned} \max_{y_1,y_2} \ y_1 + 2y_2 &\ \text{ s.t.} \\ y_2 &\leq -1, \\ -y_1 + 2y_2 &\leq 2, \\ y_1 - 2y_2 &\leq 8, \\ y_1 + y_2 &\leq 2, \\ y_1 \geq 0, y_2 &\leq 0. \end{aligned}$$

Assuming that the first constraint binds, the fourth constraint yields the following feasible dual solution: $(y_1, y_2) = (3, -1)$. It is easily checked that this attains a value of 1 and satisfies the other two constraints, which are slack. Looking for a primal solution consistent with this dual solution, by complementary slackness we must have $x_2 = x_3 = 0$ and the primal constraints binding. Therefore, it remains to solve the system

$$x_4 = 1, \qquad x_1 + x_4 = 2,$$

which leaves a feasible primal solution $(x_1, x_2, x_3, x_4) = (1, 0, 0, 1)$. Since the value of this primal feasible solution equals 1 and therefore coincides with the value of the dual feasible solution $(3, -1)$ it follows by duality that these are also optimal solutions.

4.7 The following table answers the questions:

Primal	Dual
$\max\{cx : Ax = b, x \geq 0\}$	$\min\{yb : yA \geq c\}$
$\min\{cx : Ax = b, x \geq 0\}$	$\max\{yb : yA \leq c\}$
$\max\{cx : Ax \leq b, x \geq 0\}$	$\min\{yb : yA \geq c, y \geq 0\}$
$\min\{cx : Ax \geq b, x \geq 0\}$	$\max\{yb : yA \leq c, y \geq 0\}$

4.8 Given the optimization problem

$$\min_{x,y,z} |x| + |y| + |z| \quad \text{s.t.}$$
$$x + y \ \leq \ 1$$
$$2x + z = 3$$

can be rewritten as a linear program as follows:

$$\min_{x_\pm, y_\pm, z_\pm} \ x_+ + x_- + y_+ + y_- + z_+ + z_- \ \ \text{s.t.}$$
$$x_+ - x_- + y_+ - y_- \ \leq \ 1$$
$$2x_+ - 2x_- + z_+ - z_- = 3$$
$$x_+, x_-, y_+, y_-, z_+, z_- \geq 0.$$

4.9 Given the linear program

$$V \ = \ \max_{x \geq \mathbf{0}} \sum_{j=1}^{n} c_j x_j \ \ \text{s.t.} \ \sum_{j=1}^{n} a_j x_j \ \leq \ b,$$

we must show that $V = b \max_j c_j/a_j$. This follows by duality. Indeed, the dual of this problem is given by

$$W \ = \ \min_{y \geq 0} \ yb \ \ \text{s.t.} \ \forall j \in \{1, \ldots, n\}, \ ya_j \ \geq \ c_j.$$

Notice that, since there is only one primal constraint, the dual variable y is just a scalar. Since $b > 0$ by assumption, the dual is solved by picking the smallest feasible number y. For y to be feasible it must satisfy, for every j, the inequality $ya_j \geq c_j$, which, since $a_j > 0$ by assumption, is equivalent to $y \geq c_j/a_j$. Since this must hold for every j, it must also hold for the $\widehat{j}$ with the maximum such ratio, i.e., $y \geq \max_j c_j/a_j = c_{\widehat{j}}/a_{\widehat{j}}$. Finally, since we are minimizing with respect to y, this is attained at $y = c_{\widehat{j}}/a_{\widehat{j}}$, therefore $W = V = b \max_j c_j/a_j$, as required.

4.16 We are given a zero-sum game with payoff matrix

$$A \ = \ \begin{bmatrix} 3 & -2 & 1 \\ 1 & 3 & -2 \\ -2 & 1 & 3 \end{bmatrix}$$

The symmetry of the matrix suggests that we try $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = y^*$ as a candidate equilibrium. The column player's problem is given by

$$\begin{aligned}
W = \min_{R,x_1,x_2,x_3} & \quad R \ \text{ s.t.} \\
3x_1 - 2x_2 + x_3 & \ \leq R \\
x_1 + 3x_2 - 2x_3 & \ \leq R \\
-2x_1 + x_2 + 3x_3 & \ \leq R \\
x_1 + x_2 + x_3 & \ = \ 1 \\
x_1, x_2, x_3 & \geq 0.
\end{aligned}$$

Plugging x^* into the constraints above yields a value $W \leq \frac{2}{3}$. Taking the dual yields the row player's problem.

$$\begin{aligned}
V = \max_{C,y_1,y_2,y_3} & \quad C \ \text{ s.t.} \\
3y_1 + y_2 - 2y_3 & \ \geq C \\
-2y_1 + 3y_2 + y_3 & \ \geq C \\
y_1 - 2y_2 + 3y_3 & \ \geq C \\
y_1 + y_2 + y_3 & \ = \ 1 \\
y_1, y_2, y_3 & \geq 0.
\end{aligned}$$

Plugging y^* into the dual constraints yields a value $V \geq \frac{2}{3}$. By duality, $V \leq W$, therefore $V = W$, so by duality x^* and y^* are optimal solutions, which finally implies that they form an equilibrium.

2. We are given the following linear programming problem:

$$\begin{aligned}
V \ = \ \max_{x_1,x_2} \ 3x_1 + 2x_2 & \ \text{ subject to} \\
x_1 + 2x_2 & \ \leq \ \beta, \\
2x_1 + x_2 & \ \leq \ 5, \\
x_1, x_2 & \ \geq \ 0.
\end{aligned}$$

The dual problem is given by

$$\begin{aligned}
W \ = \ \min_{y_1,y_2} \ \beta y_1 + 5y_2 & \ \text{ subject to} \\
y_1 + 2y_2 & \ \geq \ 3, \\
2y_1 + y_2 & \ \geq \ 2, \\
y_1, y_2 & \ \geq \ 0.
\end{aligned}$$

The optimal solutions are tabulated below:

β	x^*	$V = W$	y^*
$\beta < 0$	$\emptyset$	$-\infty$	unbounded (e.g., $(4n, 0)\ \forall n \in \mathbb{N}$)
$0 \leq \beta < 2.5$	$(\beta, 0)$	3β	$(3, 0)$
$\beta = 2.5$	$(\beta, 0)$	3β	$\{\lambda(3,0) + (1-\lambda)(\frac{1}{3}, \frac{4}{3}) : \lambda \in [0,1]\}$
$2.5 < \beta < 10$	$(\frac{10-\beta}{3}, \frac{2\beta-5}{3})$	$(20+\beta)/3$	$(\frac{1}{3}, \frac{4}{3})$
$\beta = 10$	$(\frac{10-\beta}{3}, \frac{2\beta-5}{3})$	$(20+\beta)/3$	$\{\lambda(\frac{1}{3}, \frac{4}{3}) + (1-\lambda)(0,2) : \lambda \in [0,1]\}$
$10 < \beta$	$(0, 5)$	10	$(0, 2)$

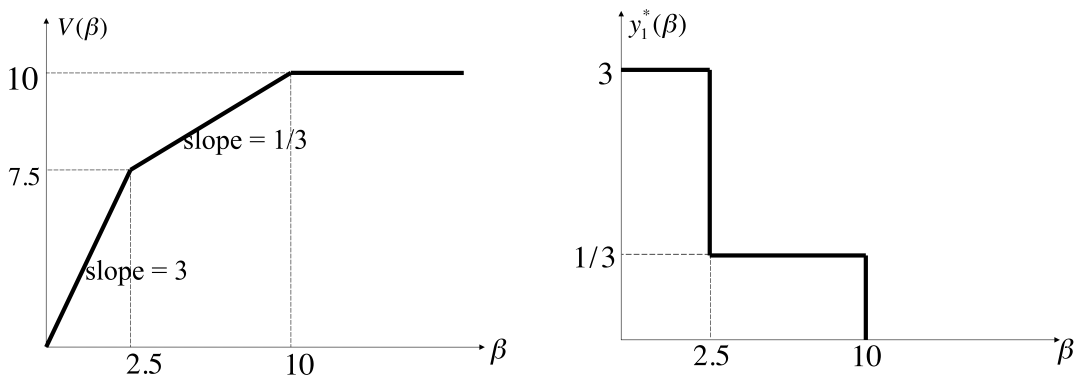

Figure 1: Plot of $V(\beta)$ and $y_1^*(\beta)$ as a function of β.

The slopes of $V(\beta)$ coincide with the optimal solutions $y_1^*(\beta)$.

Now suppose that the linear programming problem looks like this:

$$\begin{aligned} V &= \max_{x_1, x_2} \ \alpha x_1 + 2x_2 \ \text{ subject to} \\ x_1 + 2x_2 &\leq 4, \\ 2x_1 + x_2 &\leq 5, \\ x_1, x_2 &\geq 0. \end{aligned}$$

The dual problem is given by

$$\begin{aligned} W &= \min_{y_1, y_2} \ 4y_1 + 5y_2 \ \text{ subject to} \\ y_1 + 2y_2 &\geq \alpha, \\ 2y_1 + y_2 &\geq 2, \\ y_1, y_2 &\geq 0. \end{aligned}$$

The optimal solutions are tabulated below:

α	x^*	$V=W$	y^*
$\alpha < 1$	$(0,2)$	4	(1,0)
$\alpha = 1$	$\{\lambda(0,2)+(1-\lambda)(2,1) : \lambda \in [0,1]\}$	4	$(1,0)$
$1 < \alpha < 4$	$(2,1)$	$2\alpha+2$	$(\frac{4-\alpha}{3}, \frac{2(\alpha-1)}{3})$
$\alpha = 4$	$\{\lambda(2,1)+(1-\lambda)(2.5,0) : \lambda \in [0,1]\}$	10	$(\frac{4-\alpha}{3}, \frac{2(\alpha-1)}{3})$
$4 < \alpha$	$(2.5,0)$	2.5α	$(0,2)$

Now, the slopes of $V(\alpha)$ coincide with $x_1^*(\alpha)$.