Homework 3—Suggested Answers

Answers from Simon and Blume are on the back of the book. Answers to questions from Dixit's book:

2.1. We are to solve the following budget problem, where α, β, p, q, I are positive and $\alpha+\beta \leq 1$ (this assumption is necessary for concavity of the utility function):

$$\max_{x,y\geq 0}\{x^\alpha y^\beta : px + qy \leq I\}$$

The utility function is concave and the constraint function is convex (being linear) so Kuhn-Tucker conditions are necessary and sufficient for a maximum. Given the Lagrangean

$$\mathcal{L}(x, y, \lambda) = x^\alpha y^\beta + \lambda(I - px - qy),$$

the Kuhn-Tucker conditions are given by

$$\frac{\partial \mathcal{L}}{\partial x} = \alpha x^{\alpha-1} y^\beta - \lambda p \leq 0, \quad x \geq 0, \quad x(\alpha x^{\alpha-1} y^\beta - \lambda p) = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \beta x^\alpha y^{\beta-1} - \lambda q \leq 0, \quad y \geq 0, \quad y(\beta x^\alpha y^{\beta-1} - \lambda q) = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - px - qy \geq 0, \quad \lambda \geq 0, \quad \lambda(I - px - qy) = 0 \tag{3}$$

Suppose that (x^*, y^*, λ^*) satisfies the Kuhn-Tucker conditions. If the budget constraint is slack then $\lambda^* = 0$, which implies that marginal utility for both x and y is less than or equal to zero. This is only possible if both x^* and y^* equal zero, by the non-negativity constraints. However, since $I > 0$, there exists a feasible solution (x, y) such that both x and y are positive, yielding a higher utility than zero. This contradicts optimality of (x^*, y^*, λ^*), therefore the budget constraint must bind.

Both x^* and y^* must be positive. Indeed, if not then $x^* = 0$ or $y^* = 0$. (We showed in the previous paragraph that both x^* and y^* equal to zero is impossible.) Since both α and β are less than 1, marginal utility is infinity for both goods, so the Kuhn-Tucker conditions are violated (because $\infty \not\leq \lambda p$ and similarly for λq). Therefore, (x^*, y^*, λ^*) has both x^* and y^* positive.

Therefore, the FOC (1), (2) and (3) hold with equality. We must now solve

$$\alpha x^{\alpha-1} y^\beta = \lambda p \tag{4}$$

$$\beta x^\alpha y^{\beta-1} = \lambda q \tag{5}$$

$$px + qy = I \tag{6}$$

Dividing (4) by (5) and rearranging, we obtain

$$\frac{\alpha x^{\alpha-1}y^{\beta}}{\beta x^{\alpha}y^{\beta-1}} = \frac{\lambda p}{\lambda q} = \frac{p}{q} \qquad \Rightarrow \qquad \frac{\alpha y}{\beta x} = \frac{p}{q} \qquad \Rightarrow \qquad \frac{px}{qy} = \frac{\alpha}{\beta}.$$

The last equation says that the optimal expenditure ratio is α/β. Substituting into the budget constraint, we obtain

$$\begin{aligned} px + qy = I \qquad &\Rightarrow \qquad px + px\frac{\beta}{\alpha} = I \qquad \Rightarrow \qquad px\left(1 + \frac{\beta}{\alpha}\right) = I \\ &\Rightarrow \qquad px = \frac{I}{1+\frac{\beta}{\alpha}} = \frac{\alpha I}{\alpha+\beta} \qquad \Rightarrow \qquad qy = \frac{\beta I}{\alpha+\beta} \\ &\Rightarrow \qquad x^* = \frac{\alpha I}{p(\alpha+\beta)} \qquad \Rightarrow \qquad y^* = \frac{\beta I}{q(\alpha+\beta)} \end{aligned}$$

The multiplier λ^* can be found by plugging x^* and y^* into either (4) or (5)

$$\alpha x^{*\alpha-1}y^{*\beta} = \lambda p \qquad \Rightarrow \qquad \alpha\left[\frac{\alpha I}{p(\alpha+\beta)}\right]^{\alpha-1}\left[\frac{\beta I}{q(\alpha+\beta)}\right]^{\beta} = \lambda^* p$$

Rearranging, we obtain the following solution for λ^*:

$$\lambda^* = \frac{\alpha^{\alpha}\beta^{\beta}I^{\alpha+\beta-1}}{p^{\alpha}q^{\beta}(\alpha+\beta)^{\alpha+\beta}}$$

2.2. We are to solve the following budget problem:

$$\max_{x,y\geq 0}\{\alpha\ln(x-x_0) + \beta\ln(y-y_0) : px + qy \leq I\}$$

We take as given the constants x_0 and y_0 which we will assume to be positive, as well as the usual α, β, p, q and I, all positive. We assume that $\alpha+\beta = 1$. The Lagrangean of this problem is given by:

$$\mathcal{L}(x,y,\lambda) = \alpha\ln(x-x_0) + \beta\ln(y-y_0) + \lambda(I - px - qy)$$

The Kuhn-Tucker FOC are given by:

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{\alpha}{x-x_0} - \lambda p \leq 0, \quad x \geq 0, \quad x\left(\frac{\alpha}{x-x_0} - \lambda p\right) = 0 \tag{7}$$

$$\frac{\partial\mathcal{L}}{\partial y} = \frac{\beta}{y-y_0} - \lambda q \leq 0, \quad y \geq 0, \quad y\left(\frac{\beta}{y-y_0} - \lambda q\right) = 0 \tag{8}$$

$$\frac{\partial\mathcal{L}}{\partial \lambda} = I - px - qy \geq 0, \quad \lambda \geq 0, \quad \lambda(I - px - qy) = 0 \tag{9}$$

The natural logarithm is not defined for non-positive values, so necessarily $x > x_0$ and $y > y_0$. Since x_0 and y_0 are both positive and marginal utility is always positive, it follows that all three first-order conditions hold with equality.

Hence, we must solve the following system of equations:

$$\frac{\alpha}{x - x_0} = \lambda p \tag{10}$$

$$\frac{\beta}{y - y_0} = \lambda q \tag{11}$$

$$px + qy = I \tag{12}$$

Dividing (10) by (11) and rearranging, we obtain

$$\frac{\alpha(y - y_0)}{\beta(x - x_0)} = \frac{\lambda p}{\lambda q} = \frac{p}{q} \qquad \Rightarrow \qquad \frac{\alpha}{\beta} = \frac{p(x - x_0)}{q(y - y_0)}$$

$$\Rightarrow \qquad \alpha q(y - y_0) = \beta p(x - x_0) \qquad \Rightarrow \qquad \alpha q y = \beta p(x - x_0) + q y_0$$

$$\Rightarrow \qquad \beta p x = \alpha q(y - y_0) + p x_0.$$

Plugging this into the budget constraint and using $\alpha + \beta = 1$ yields

$$\alpha I = \alpha p x + \alpha q y = \alpha p x + \beta p(x - x_0) + q y_0 \qquad \Rightarrow \qquad p x^* = \alpha I + \beta p x_0 - q y_0$$

$$\Rightarrow \qquad q y^* = \beta I + \alpha q y_0 - p x_0.$$

2.3. We are asked to minimize cost, which in this case amounts to expenditure on inputs, subject to a production constraint. The optimization problem is:

$$C(Q) = \min_{K,L \geq 0} \{rK + wL : \sqrt{K} + \sqrt{L} \geq Q\}$$

The Lagrangean is given by:

$$\mathcal{L}(x, y, \lambda) = -rK - wL + \lambda(\sqrt{K} + \sqrt{L} - Q)$$

The Kuhn-Tucker conditions are given by

$$\frac{\partial \mathcal{L}}{\partial K} = -r + \tfrac{1}{2}\lambda K^{-1/2} \leq 0, \quad K \geq 0, \quad K\left(-r + \tfrac{1}{2}\lambda K^{-1/2}\right) = 0 \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial L} = -w + \tfrac{1}{2}\lambda L^{-1/2} \leq 0, \quad L \geq 0, \quad L\left(-w + \tfrac{1}{2}\lambda L^{-1/2}\right) = 0 \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sqrt{K} + \sqrt{L} - Q \geq 0, \quad \lambda \geq 0, \quad \lambda(\sqrt{K} + \sqrt{L} - Q) = 0 \tag{15}$$

Since the objective is concave (being linear) and the constraint function is convex ($Q - \sqrt{K} - \sqrt{L} \leq 0$) any solution to the Kuhn-Tucker conditions solves the optimization. By (15), we must have either K or L or both positive. Therefore, if K is positive, say, then (13) must hold with equality, so $\lambda > 0$. If L equals zero then (14) is violated. Similarly, we get a contradiction if K is zero and L is positive. Hence, both K and L must be positive with $\lambda > 0$, so all three conditions above hold with equality.

We must therefore solve the following three equations:

$$r = \tfrac{1}{2}\lambda K^{-1/2} \tag{16}$$

$$w = \tfrac{1}{2}\lambda L^{-1/2} \tag{17}$$

$$\sqrt{K} + \sqrt{L} = Q \tag{18}$$

Dividing (16) by (17), we obtain

$$\frac{r}{w} = \frac{K^{-1/2}}{L^{-1/2}} = \left(\frac{L}{K}\right)^{1/2} \qquad \Rightarrow \qquad \frac{L}{K} = \left(\frac{r}{w}\right)^2.$$

Substituting into the production constraint yields

$$\sqrt{K} + \sqrt{L} = \sqrt{K} + \sqrt{K\left(\frac{r}{w}\right)^2} = \left(\frac{r}{w} + 1\right)\sqrt{K} = Q \qquad \Rightarrow \qquad K = \left[\frac{Q}{\frac{r}{w} + 1}\right]^2$$

$$\Rightarrow \qquad K^* = \left[\frac{wQ}{r+w}\right]^2 \qquad \Rightarrow \qquad L^* = \left[\frac{rQ}{r+w}\right]^2$$

By (16), the Lagrange multiplier is given by

$$\lambda^* = 2r\sqrt{K^*} = \frac{2rwQ}{r+w}$$

The quantity λ^* is the *marginal cost* of production at Q.

Now we are given that the price of output equals p, and that output is a choice. The firm's profit-maximization problem is:

$$\max_{K,L,Q\geq 0} \{pQ - rK - wL : \sqrt{K} + \sqrt{L} \geq Q\}$$

Since it would not be profit-maximizing to waste inputs, we may assume that the production constraint holds with equality. Then we may write directly the optimization problem as:

$$\max_{K,L\geq 0} \{p(\sqrt{K} + \sqrt{L}) - rK - wL\}$$

This is a concave function of K and L, so FOC are necessary and sufficient for a maximum. The two conditions are:

$$\tfrac{1}{2}pK^{-1/2} = r \qquad \Rightarrow \qquad K^* = \left(\frac{2r}{p}\right)^{-2} \tag{19}$$

$$\tfrac{1}{2}pL^{-1/2} = w \qquad \Rightarrow \qquad L^* = \left(\frac{2w}{p}\right)^{-2} \tag{20}$$

Therefore,

$$Q^* = \sqrt{K^*} + \sqrt{L^*} = \frac{p}{2r} + \frac{p}{2w} = \frac{pw}{2rw} + \frac{pr}{2rw} = \frac{p(r+w)}{2rw}.$$

Since the firm takes output prices as given, average revenue—or *price*—equals marginal revenue. Optimizing behavior implies that the firm equates marginal revenue with marginal cost, i.e.,

$$p = \lambda^* \qquad \Rightarrow \qquad p = \frac{2rwQ^*}{r+w} \qquad \Rightarrow \qquad Q^* = \frac{p(r+w)}{2rw}.$$

3.1. Let us first solve the budget problem without a rationing constraint:

$$\max_{x_1,x_2,x_3\geq 0}\{\alpha_1\ln(x_1) + \alpha_2\ln(x_2) + \alpha_3\ln(x_3) : p_1x_1 + p_2x_2 + p_3x_3 \leq I\}$$

We are given that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. The Lagrangean is given by:

$$\mathcal{L}(x_1, x_2, x_3, \lambda) = \alpha_1\ln(x_1) + \alpha_2\ln(x_2) + \alpha_3\ln(x_3) + \lambda(I - p_1x_1 - p_2x_2 - p_3x_3)$$

The utility function is concave, so a Kuhn-Tucker vector is necessary and sufficient for a maximum. By the same arguments as in question 2.1, $x_i^* > 0$ and $\lambda^* > 0$, so all FOCs hold with equality. Therefore, finding a Kuhn-Tucker point boils down to solving the following equations:

$$\frac{\alpha_1}{x_1} = \lambda p_1 \tag{21}$$

$$\frac{\alpha_2}{x_2} = \lambda p_2 \tag{22}$$

$$\frac{\alpha_3}{x_3} = \lambda p_3 \tag{23}$$

$$p_1x_1 + p_2x_2 + p_3x_3 = I \tag{24}$$

Equations (21–23) imply that $x_i^* = \alpha_i/(\lambda^* p_i)$. Substituting this into the budget constraint yields

$$p_1x_1 + p_2x_2 + p_3x_3 = I \qquad \Rightarrow \qquad \frac{\alpha_1}{\lambda^*} + \frac{\alpha_2}{\lambda^*} + \frac{\alpha_3}{\lambda^*} = I \qquad \Rightarrow \qquad \lambda^* = 1/I$$

Therefore, $x_i^* = \alpha_i I/p_i$ and $p_i x_i/I = \alpha_i$. In other words, the consumer's optimal expenditure share of income on good i equals α_i.

Now consider the rationing problem. The new maximization problem looks like:

$$\max_{x_1,x_2,x_3 \geq 0} \{\alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) + \alpha_3 \ln(x_3) : p_1x_1 + p_2x_2 + p_3x_3 \leq I,\ x_1 \leq k\}$$

Clearly, if $k \geq \alpha_i I/p_i$ then the consumer can purchase his optimal bundle as if the rationing constraint were not there, which must be optimal. On the other hand, if $k < \alpha_i I/p_i$ then the rationing constraint will bind. Therefore, $x_1^{**} = k$. (We will use ** to denote optimal solutions of the rationed problem.) Denote by μ the multiplier on the rationing constraint. The Lagrangean looks like:

$$\mathcal{L} = \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) + \alpha_3 \ln(x_3) + \lambda(I - p_1x_1 - p_2x_2 - p_3x_3) + \mu(k - x_1)$$

By the same arguments as before, all the FOCs will hold with equality, so we are left with solving:

$$\frac{\alpha_1}{x_1} = \lambda p_1 + \mu \tag{25}$$
$$\frac{\alpha_2}{x_2} = \lambda p_2 \tag{26}$$
$$\frac{\alpha_3}{x_3} = \lambda p_3 \tag{27}$$
$$p_1x_1 + p_2x_2 + p_3x_3 = I \tag{28}$$
$$x_1 = k \tag{29}$$

Substituting the last equation and defining $\widehat{I} = I - p_1 k$ implies

$$\frac{\alpha_2}{x_2} = \lambda p_2 \tag{30}$$
$$\frac{\alpha_3}{x_3} = \lambda p_3 \tag{31}$$
$$p_2x_2 + p_3x_3 = \widehat{I} \tag{32}$$

Dividing (30) by (31) yields

$$\frac{\alpha_2 x_3}{\alpha_3 x_2} = \frac{p_2}{p_3} \qquad \Rightarrow \qquad \frac{p_2x_2}{p_3x_3} = \frac{\alpha_2}{\alpha_3}.$$

Therefore, the ratio of expenditure on good 2 relative to good 3 equals α_2/α_3. Substituting this ratio into (32), we obtain

$$p_2x_2 + p_3x_3 = \widehat{I} \qquad \Rightarrow \qquad \frac{\alpha_2}{\alpha_3}p_3x_3 + p_3x_3 = \widehat{I} \qquad \Rightarrow \qquad \frac{\alpha_2+\alpha_3}{\alpha_3}p_3x_3 = \widehat{I}$$
$$\Rightarrow \qquad p_3x_3^* = \frac{\alpha_3\widehat{I}}{\alpha_2+\alpha_3}$$
$$\Rightarrow \qquad p_2x_2^* = \frac{\alpha_2\widehat{I}}{\alpha_2+\alpha_3}$$

Notice that rationing of good 1 implies that demand for the other goods increases, so goods are substitutes. However, if bread and butter are complements then rationing bread diminishes demand for butter.

3.2. We are to solve the following social planning problem:

$$\max_{y_1,y_2}\{y_1 - ky_2^2 + y_2 - ky_1^2 : y_1 + y_2 \leq Y\}$$

We are given that $k > 0$. Suppose that $Y > 1/k$. We must show that the resource constraint is slack at the optimum. The Lagrangean is given by

$$\mathcal{L} = y_1 - ky_2^2 + y_2 - ky_1^2 + \lambda(Y - y_1 - y_2)$$

The Kuhn-Tucker conditions are

$$\frac{\partial \mathcal{L}}{\partial y_1} = 1 - 2ky_1 - \lambda = 0 \tag{33}$$

$$\frac{\partial \mathcal{L}}{\partial y_2} = 1 - 2ky_2 - \lambda = 0 \tag{34}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = Y - y_1 - y_2 \geq 0, \quad \lambda \geq 0, \quad \lambda(Y - y_1 - y_2) = 0 \tag{35}$$

Notice that $\lambda^* = 0$ together with $y_1^* = 1/(2k) = y_2^*$ solves the system above. Since the objective is concave this Kuhn-Tucker point solves the problem.

The interpretation is that envy implies that eventually the goods become bads, in that more of the good (beyond $1/(2k)$ destroys value due to envy.

3.3. The optimization problem is:

$$\max_{x_1,\ldots,x_n \geq 0}\left\{\sum_{j=1}^{n} \alpha_j x_j - \tfrac{1}{2}\beta_j x_j^2 : \sum_{j=1}^{n} x_j \leq C\right\}$$

We assume that $\alpha_j, \beta_j, C > 0$. The Lagrangean is given by:

$$\mathcal{L} = \sum_{j=1}^{n} \alpha_j x_j - \tfrac{1}{2}\beta_j x_j^2 + \lambda\left[C - \sum_{j=1}^{n} x_j\right]$$

The objective is concave, so the Kuhn-Tucker conditions are necessary and sufficient to solve the problem. The Kuhn-Tucker conditions are:

$$\forall j, \quad \frac{\partial \mathcal{L}}{\partial x_j} = \alpha_j - \beta_j x_j - \lambda \leq 0, \quad x_j \geq 0, \quad x_j(\alpha_j - \beta_j x_j - \lambda) = 0 \tag{36}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = C - \sum_{i=1}^{n} x_j \geq 0, \quad \lambda \geq 0, \quad \lambda\left[C - \sum_{j=1}^{n} x_j\right] = 0 \tag{37}$$

Define

$$H = \sum_{j=1}^{n} \alpha_j/\beta_j, \qquad K = \sum_{j=1}^{n} 1/\beta_j.$$

(i) Suppose that $C > H$. In this case, notice that by (36), we have that

$$\alpha_j - \beta_j x_j \le \lambda \qquad \Rightarrow \qquad x_j \ge (\alpha_j - \lambda)/\beta_j.$$

Letting $\lambda^* = 0$ and $x_j^* = \alpha_j/\beta_j$ satisfies (36) for every j. Furthermore, since $C > H$, it follows that $C - \sum_j x_j^* > 0$, so (37) is satisfied, too. Therefore, this is an optimal solution to the problem. Since $\sum_j x_j^* < C$, a part of the total sum available is left unused.

(ii) Suppose that $\alpha_j > (H - C)/K$ for every j. By the Kuhn-Tucker Theorem, if we find a solution to the Kuhn-Tucker conditions then it will solve the optimization problem. Firstly, notice that $\sum x_j^* > 0$, since otherwise this would imply that $\lambda^* = 0$ and $x_j^* = 0$ for every j, which would not be a Kuhn-Tucker solution. If $x_j^* > 0$ for all j then rearranging (36) and adding with respect to j, we obtain

$$\frac{\lambda}{\beta_j} = \frac{\alpha_j}{\beta_j} - x_j \qquad \Rightarrow \qquad \sum_{j=1}^{n} \frac{\lambda}{\beta_j} = \sum_{j=1}^{n} \frac{\alpha_j}{\beta_j} - x_j \qquad \Rightarrow \qquad \lambda K = H - \sum_{j=1}^{n} x_j.$$

The resource constraint $C \ge \sum_j x_j$ implies that $\lambda \le (H - C)/K$.

Let $\lambda^* = (H - C)/K$ and choose x_j^* so that

$$\lambda^* = \alpha_j - \beta_j x_j^*.$$

Since $\alpha_j > (H - C)/K$ for every j, it follows that $x_j^* = (\alpha_j - \lambda)/\beta_j > 0$. Finally,

$$\sum_{j=1}^{n} x_j^* = \sum_{j=1}^{n} \frac{\alpha_j - \lambda}{\beta_j} = H - (H - C)K/K = C.$$

Therefore, $(x_1^*, \ldots, x_n^*, \lambda^*)$ is a Kuhn-Tucker vector, since it satisfies all the conditions in (36) and (37).

(iii) Suppose that project j_0 receives zero funding but j_1 receives some funding. By (36), it follows that $\alpha_{j_0} \le \lambda$ and $\alpha_{j_1} - \beta_{j_1} x_{j_1} = \lambda$. Therefore,

$$\alpha_{j_0} = \alpha_{j_1} - \beta_{j_1} x_{j_1} < \alpha_{j_1},$$

since $x_{j_1} > 0$.