Homework 4—Suggested Answers

Answers from Simon and Blume are on the back of the book. Answers to questions from Varian's chapter:

17.4. By free disposal, all prices must be non-negative. To calculate market-clearing prices, notice that if both prices are positive then B will consume $x_1^B = x_2^B$, i.e., the same amount of both goods. If one price is negative and the other equals zero then one consumer will buy some amount of the free good and the other consumer will not be able to afford anything, so the market will not clear. Also, if both prices equal zero then both consumers will demand an unbounded amount, which clearly fails to clear the market. Therefore, both prices must be strictly positive, so $x_1^B = x_2^B$. By market-clearing, $x_1^A + x_1^B = \omega_1^A + \omega_1^B = 1$ and $x_2^A + x_2^B = \omega_2^A + \omega_2^B = 1$. Hence, $x_1^A = x_2^A$. To pin down (relative) prices, notice that these must be equated to players' marginal rates of substitution. Looking at consumer A's budget problem, we find that his first-order conditions lead to

$$\frac{ax_2^A}{(1-a)x_1^A} = \frac{p_1}{p_2}.$$

Since A consumes as much of good 1 as of good 2, $p_1/p_2 = a/(1-a)$. This pins down the price as much as possible, since demand curves are homogeneous of degree zero. As regards equilibrium allocations, these are obtained from the budget constraints, which will bind because individuals have strictly increasing utility functions. Therefore, we are left with solving the following two equations in two unknowns:

$$\begin{aligned} p_1x^A + p_2x^A &= p_2 \\ p_1x^B + p_2x^B &= p_1 \end{aligned}$$

where $x^A = x_1^A = x_2^A$ and $x^B = x_1^B = x_2^B$. Substituting the relative prices obtained previously, it follows that

$$\begin{aligned} \frac{a}{1-a}p_2x^A + p_2x^A &= p_2 \\ \Rightarrow \quad (\frac{a}{1-a} + 1)x^A &= 1 \\ \Rightarrow \quad x^A &= 1-a \\ \Rightarrow \quad x^B &= a. \end{aligned}$$

17.11. The Edgeworth box looks like Figure 1 below. The straight line denotes A's indifference curve and the "L" denotes B's indifference curve. To understand equilibrium, notice that every individual can afford not to trade. Therefore, the equilibrium price line must always pass through the endowment point $(1/2, 1/2)$.

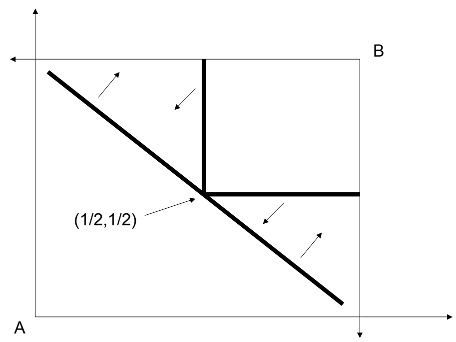

Figure 1: Edgeworth Box.

Notice that if prices are not equal then A will demand more of the cheaper good than is available. To see this, let $p_1 > p_2$ and consider A's budget problem:

$$\max_{x_1^A, x_2^A \geq 0} \{x_1^A + x_2^A \ : \ p_1 x_1^A + p_2 x_2^A \leq p_1 \tfrac{1}{2} + p_2 \tfrac{1}{2}\}$$

This is a linear programming problem. Its dual is given by

$$\min_{\lambda \geq 0} \tfrac{1}{2}\{\lambda(p_1 + p_2) \ : \ \lambda p_1 \geq 1, \ \lambda p_2 \geq 1\}$$

Since $p_1 > p_2$, it follows that $\lambda p_2 \geq 1$ implies that $\lambda p_1 > 1$, so only the second constraint will bind. Therefore, by complementary slackness, consumer A will only consume a positive amount of good 2, the cheaper good. Since utility is strictly increasing, A will end up spending all of $p_1 \frac{1}{2} + p_2 \frac{1}{2}$ on good 2. Therefore,

$$x_2^A \ = \ \frac{p_1 \frac{1}{2} + p_2 \frac{1}{2}}{p_2} \ > \ \frac{p_2 \frac{1}{2} + p_2 \frac{1}{2}}{p_2} \ = \ 1,$$

but this amount is larger than the amount of good 2 available in the economy. Therefore, markets cannot clear if $p_1 > p_2$. A symmetric argument implies that markets cannot clear if $p_1 < p_2$. Therefore, equilibrium requires that $p_1 = p_2$.

Since equilibrium prices are equal, consumer A is indifferent between any allocation on the budget line. On the other hand, consumer B has a preference for extremes, so will maximize utility by spending all income on just one good. Either one will do. Hence, there are two equilibrium allocations: $x_1^A = 1 = x_2^B$, $x_1^B = 0 = x_2^A$, and $x_2^A = 1 = x_1^B$, $x_2^B = 0 = x_1^A$.

9.1. Terminal wealth is given by $W = W_0 + (1-\tau)xr$, where r is a random variable. Expected utility over terminal wealth is given by

$$V(x) \;=\; \int_{\underline{r}}^{\overline{r}} U(W_0 + (1-\tau)xr) f(r) dr.$$

The first-order condition for an interior optimum is given by

$$V_x \;=\; (1-\tau) \int_{\underline{r}}^{\overline{r}} rU'(W_0 + (1-\tau)xr) f(r) dr \;=\; 0.$$

Dividing both sides by $(1-\tau)$, we obtain the condition

$$\int_{\underline{r}}^{\overline{r}} rU'(W_0 + (1-\tau)xr) f(r) dr \;=\; 0.$$

If τ changes, the investor will change x optimally, i.e., so as to keep $V_x = 0$, where now it is acknowledged that V_x depends both on x and τ. Totally differentiating the first-order conditions on both sides yields

$$V_{x\tau} d\tau + V_{xx} dx \;=\; 0 \qquad \Rightarrow \qquad \frac{dx}{d\tau} \;=\; -\frac{V_{x\tau}}{V_{xx}}.$$

Differentiating V_x with respect to τ yields

$$\begin{aligned} V_{x\tau} \;&=\; -(1-\tau)x \int_{\underline{r}}^{\overline{r}} r^2 U''(W_0 + (1-\tau)xr) f(r) dr \\ &\qquad - \int_{\underline{r}}^{\overline{r}} rU'(W_0 + (1-\tau)xr) f(r) dr \\ &=\; -(1-\tau)x \int_{\underline{r}}^{\overline{r}} r^2 U''(W_0 + (1-\tau)xr) f(r) dr, \end{aligned}$$

where the second equality follows by the first-order condition. Similarly,

$$V_{xx} \;=\; (1-\tau)^2 \int_{\underline{r}}^{\overline{r}} r^2 U''(W_0 + (1-\tau)xr) f(r) dr.$$

Therefore, the derivative $dx/d\tau$ is given by

$$\frac{dx}{d\tau} \;=\; -\frac{-(1-\tau)x \int_{\underline{r}}^{\overline{r}} r^2 U''(W_0 + (1-\tau)xr) f(r) dr}{(1-\tau)^2 \int_{\underline{r}}^{\overline{r}} r^2 U''(W_0 + (1-\tau)xr) f(r) dr} \;=\; \frac{x}{1-\tau}.$$

Let $F(x,\tau) = x(1-\tau)$. To find $dx/d\tau$ on the constraint $F(x,\tau) = C$, i.e., subject to $F(x,\tau)$ remaining at some constant C, we totally differentiate:

$$F_\tau d\tau + F_x dx \;=\; 0 \qquad \Rightarrow \qquad \frac{dx}{d\tau} \;=\; -\frac{F_\tau}{F_x} \;=\; -\frac{-x}{1-\tau} \;=\; \frac{x}{1-\tau}.$$

Therefore, the optimal response with x to a change in τ is to keep the product $x(1-\tau)$ constant.

9.2. The investor's maximization problem is to choose saving S in order to solve:

$$\max_S \; V(S) \;=\; U(Y_1 - S) + \delta E[U(Y_2 + rS)]$$

The first- and second-order conditions are given by

$$\begin{aligned} V_S &= -U'(Y_1 - S) + \delta E[rU'(Y_2 + rS)] \;=\; 0 \\ V_{SS} &= U''(Y_1 - S) + \delta E[r^2 U''(Y_2 + rS)] \;<\; 0. \end{aligned}$$

As Y_1 increases, S changes so as to keep the first-order condition satisfied, i.e.,

$$V_{SY_1} dY_1 + V_{SS} dS \;=\; 0 \qquad \Rightarrow \qquad \frac{dS}{dY_1} \;=\; -\frac{V_{SY_1}}{V_{SS}}.$$

Therefore,

$$\begin{aligned} \frac{dS}{dY_1} &= -\frac{-U''(Y_1 - S)}{U''(Y_1 - S) + \delta E[r^2 U''(Y_2 + rS)]} \\ &= \frac{-U''(Y_1 - S)}{-U''(Y_1 - S) - \delta E[r^2 U''(Y_2 + rS)]} \end{aligned}$$

Since $U'' < 0$, the numerator is clearly positive. But since $r^2 > 0$ and the expectation of a negative-valued random variable (namely $r^2 U'(Y_2 + rS)$) is negative, it follows that the denominator is negative, too. Hence, $dS/dY_1 > 0$. At the same time, since $E[r^2 U''(Y_2 + rS)] < 0$, it follows that

$$0 \;<\; -U''(Y_1 - S) \;<\; -U''(Y_1 - S) - \delta E[r^2 U''(Y_2 + rS)],$$

therefore $dS/dY_1 < 1$, too.

If Y_2 is sure but r is random, to obtain dS/dY_2 we calculate $-V_{SY_2}/V_{SS}$:

$$\frac{dS}{dY_2} \;=\; -\frac{\delta E[rU''(Y_2 + rS)]}{U''(Y_1 - S) + \delta E[r^2 U''(Y_2 + rS)]}$$

The sign of this is not determined. It could be positive or negative depending on whether $E[rU''(Y_2 + rS)]$ is negative or positive. If U'' is very negative when r is positive and not very negative when r is negative then this quantity will be negative, and vice versa.

Finally, if r is sure but Y_2 is random, to obtain dS/dr we calculate $-V_{Sr}/V_{SS}$:

$$\frac{dS}{dr} \;=\; -\frac{\delta E[rSU''(Y_2 + rS) + U'(Y_2 + rS)]}{U''(Y_1 - S) + \delta E[r^2 U''(Y_2 + rS)]}$$

Again, the sign of this derivative is indeterminate except for when the optimal amount of saving and or return is either relatively very small or very large. If it is very small then then the sign of the derivative is negative. Otherwise, if it is very large then the sign of the derivative will be positive.

10.1. Solving the problem of Example 10.1 with $U(c) = c^{1-\epsilon}/(1-\epsilon)$ amounts to finding the optimal consumption path. The capital accumulation equation stays the same:

$$\dot{k} \;=\; w + rk - c$$

The Hamiltonian becomes

$$H \;=\; U(c)e^{-\rho t} + \pi[w + rk - c] \;=\; \frac{c^{1-\epsilon}}{1-\epsilon}e^{-\rho t} + \pi[w + rk - c].$$

The first-order condition for c to maximize H is

$$c^{-\epsilon}e^{-\rho t} \;=\; \pi \qquad \Rightarrow \qquad c = \left[\frac{e^{-\rho t}}{\pi}\right]^{1/\epsilon}$$

The differential equation satisfied by π is still

$$\dot{\pi} \;=\; -\frac{\partial H^*}{\partial k} \;=\; -r\pi,$$

which is solved by $\pi(t) \;=\; \pi(0)e^{-rt}$. (Here, H^* is the value function of the Hamiltonian.) The differential equation for k is

$$\dot{k} \;=\; \frac{\partial H^*}{\partial \pi} \;=\; w + rk - \frac{e^{-\rho t}}{\pi}$$

Substituting the solution for π, we obtain

$$\dot{k} \;=\; w + rk - e^{(r-\rho)t}$$

Just as in Example 10.1, we obtain from here that

$$k(T)e^{-rT} - k(0) \;=\; \frac{w(1-e^{-rT})}{r} - \frac{1-e^{-\rho t}}{\pi_0 \rho}.$$

Therefore,

$$\pi(0) \;=\; \frac{r(1-e^{-\rho t})}{\rho[w(1-e^{-rT}) - r(k(T)e^{-rT} - k(0))]}.$$

The problem becomes infeasible if

$$w(1-e^{-rT}) - r(k(T)e^{-rT} - k(0)) \;<\; 0,$$

i.e., if $k(T) > e^{rt}[w(1-e^{-rT})/r + k(0)]$.

10.2. We are to derive the differential equations below:

$$\begin{aligned}\dot{k} \;&=\; F(k) - \delta k - G(\varphi)\\ \dot{\varphi} \;&=\; -\varphi[F'(k) - \rho - \delta]\end{aligned}$$

The variable φ is defined in Example 10.2 as follows: $\varphi = \pi e^{\rho t}$. Since by the first-order conditions for the Hamiltonian we obtain

$$U'(c)e^{-\rho t} \;=\; \pi$$

it follows that $\varphi = U'(c)$ is the marginal utility of consumption. Let G be the inverse function for U'. That is, $G(\varphi) = c$ by definition if $\varphi = U'(c)$. Therefore we obtain the first differential equation by substituting c for $G(\varphi)$, i.e., $\dot{k} = F(k) - \delta k - G(\varphi)$. For the second equation, recall the equation derived in Example 10.2, namely

$$\frac{\dot{c}}{c} \;=\; \frac{F'(k) - (\rho + \delta)}{\eta(c)},$$

where $\eta(c) = -cU''(c)/U'(c)$. Canceling out c yields

$$\dot{c} \;=\; -\frac{F'(k) - (\rho + \delta)}{U''(c)/U'(c)}.$$

Rearranging,

$$U''(c)\dot{c} \;=\; -U'(c)[F'(k) - (\rho + \delta)].$$

Finally, notice that $\varphi = U'(c)$ implies that $\dot{\varphi} = U''(c)\dot{c}$, and $U'(c) = \varphi$, hence

$$\dot{\varphi} \;=\; -\varphi[F'(k) - (\rho + \delta)].$$