Midterm 1—Suggested Answers

1. (a) Given the following linear program,

$$\begin{aligned} \min_{\pi \in \mathbb{R}^3} \ \pi_1 + \pi_2 + \pi_3 & \ \text{ subject to} \\ \pi_1 + \pi_2 &\geq 1, \\ \pi_1 + \pi_3 &\geq 1, \\ \pi_2 + \pi_3 &\geq 1, \\ \pi_1, \pi_2, \pi_3 &\geq 0, \end{aligned}$$

its dual is given by

$$\begin{aligned} \max_{\sigma \in \mathbb{R}^3} \ \sigma_1 + \sigma_2 + \sigma_3 & \ \text{ subject to} \\ \sigma_1 + \sigma_2 &\leq 1, \\ \sigma_1 + \sigma_3 &\leq 1, \\ \sigma_2 + \sigma_3 &\leq 1, \\ \sigma_1, \sigma_2, \sigma_3 &\geq 0. \end{aligned}$$

Adding the primal constraints, it follows that $\pi_1 + \pi_2 + \pi_3 \geq 1.5$. Adding the dual constraints, it follows that $\sigma_1 + \sigma_2 + \sigma_3 \leq 1.5$. The vector $\pi = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a feasible primal solution with value 1.5, and the vector $\sigma = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a feasible dual solution with value 1.5. By duality, $1.5 \leq V = W \leq 1.5$, therefore π is an optimal primal solution and σ is an optimal dual solution.

(b) If $\alpha \leq \frac{3}{2}$ then the additional constraint $\pi_1 + \pi_2 + \pi_3 \geq \alpha$ doesn't affect the problem, so the solution is the same as before. If $\alpha > \frac{3}{2}$ then the value of the problem equals α with an optimal solution $\pi = (\frac{\alpha}{3}, \frac{\alpha}{3}, \frac{\alpha}{3})$. This is clearly feasible, since $\frac{\alpha}{3} + \frac{\alpha}{3} = \frac{2\alpha}{3} > 1$.

2. (a) We are given the linear program:

$$\begin{aligned} V(\alpha) \ = \ \max_{x_1, x_2} \ x_1 + 2x_2 & \ \text{ subject to} \\ x_1 + \tfrac{8}{3}x_2 &\leq 2\alpha, \\ x_1 + x_2 &\leq \alpha, \\ 2x_1 \quad &\leq 3 \\ x_1, x_2 &\geq 0. \end{aligned}$$

The dual of this problem is given below:

$$
\begin{aligned}
W(\alpha) \;=\; \min_{y_1,y_2,y_3} \; 2\alpha y_1 + \alpha y_2 + 3y_3 \;\; &\text{subject to} \\
y_1 + y_2 + 2y_3 \;&\geq\; 1, \\
\tfrac{8}{3}y_1 + y_2 \;&\geq\; 2, \\
y_1, y_2 \;&\geq\; 0.
\end{aligned}
$$

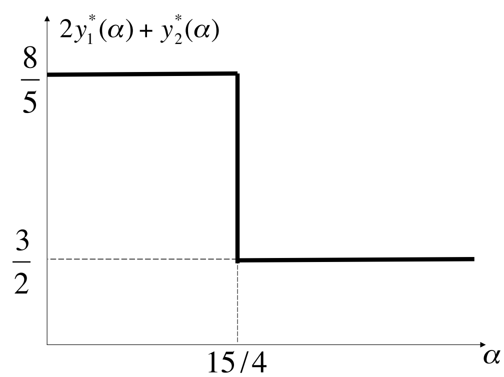
Since the slope of the isovalue line lies between the slopes of the first and second constraints, the optimal solution will solve

$$
\begin{aligned}
x_1 + \tfrac{8}{3}x_2 &= 2\alpha \\
x_1 + x_2 &= \alpha,
\end{aligned}
$$

yielding $(x_1^*, x_2^*) = (\frac{2}{5}\alpha, \frac{3}{5}\alpha)$, as long as $x_1^* \leq 3/2$, i.e., as long as $\alpha \leq 15/4$. If $\alpha > 15/4$ then $(x_1^*, x_2^*) = (\frac{3}{2}, \frac{4\alpha-3}{16})$. The remaining optimal solutions as well as the value are tabulated below.

α	x^*	$V = W$	y^*	$2y_1^* + y_2^*$
$\alpha < 0$	$\emptyset$	$-\infty$	unbounded	unbdd
$\alpha = 0$	$(0, 0)$	0	any feasible $(y_1, y_2, 0)$	$+\infty$
$0 < \alpha < 15/4$	$(\frac{2}{5}\alpha, \frac{3}{5}\alpha)$	$\frac{8}{5}\alpha$	$(\frac{3}{5}, \frac{2}{5}, 0)$	$\frac{8}{5}$
$\alpha = 15/4$	$(\frac{3}{2}, \frac{9}{4})$	6	$\{\lambda(\frac{3}{5}, \frac{2}{5}, 0) + (1-\lambda)(\frac{3}{4}, 0, \frac{1}{8}) : \lambda \in [0,1]\}$	$[\frac{3}{2}, \frac{8}{5}]$
$15/4 < \alpha$	$(\frac{3}{2}, \frac{12\alpha-9}{16})$	$\frac{3}{2}\alpha + \frac{3}{8}$	$(\frac{3}{4}, 0, \frac{1}{8})$	$\frac{3}{2}$

Therefore, $2y_1^* + y_2^*$ coincides with the slope of $V(\alpha)$.



The remaining graphs can be drawn from the table above.

(b) Now suppose that $\alpha = 1$ is fixed, and vary β in the linear program below.

$$\begin{aligned} V(\alpha) &= \max_{x_1,x_2} \; x_1 + 2x_2 \;\; \text{subject to} \\ & \qquad x_1 + \tfrac{8}{3}x_2 \leq 2, \\ & \qquad x_1 + x_2 \leq 1+\beta, \\ & \qquad 2x_1 \leq 3+\beta, \\ & \qquad x_1, x_2 \geq 0. \end{aligned}$$

The dual of this problem is given below:

$$\begin{aligned} W(\alpha) &= \min_{y_1,y_2,y_3} \; 2y_1 + (1+\beta)y_2 + (3+\beta)y_3 \;\; \text{subject to} \\ & \qquad y_1 + y_2 + 2y_3 \geq 1, \\ & \qquad \tfrac{8}{3}y_1 + y_2 \geq 2, \\ & \qquad y_1, y_2 \geq 0. \end{aligned}$$

Using the same techniques as the previous problem leads to the following table:

β	x^*	$V = W$	y^*	$y_2^* + y_3^*$
$\beta < -1$	$\emptyset$	$-\infty$	unbounded	unbdd
$\beta = -1$	$(0,0)$	0	unbounded	unbdd
$-1 < \beta < -\frac{1}{4}$	$(0, 1+\beta)$	$2(1+\beta)$	$(0,2,0)$	2
$\beta = -\frac{1}{4}$	$(0, \frac{3}{4})$	$\frac{3}{2}$	$\{\lambda(0,2,0) + (1-\lambda)(\frac{3}{5}, \frac{2}{5}, 0) : \lambda \in [0,1]\}$	$[\frac{2}{5}, 2]$
$-\frac{1}{4} < \beta < 1$	$(\frac{2+8\beta}{5}, \frac{3(1-\beta)}{5})$	$\frac{8+2\beta}{5}$	$(\frac{3}{5}, \frac{2}{5}, 0)$	$\frac{2}{5}$
$\beta = 1$	$(2,0)$	2	$\{\lambda(\frac{3}{5}, \frac{2}{5}, 0) + (1-\lambda)(1,0,0) : \lambda \in [0,1]\}$	$[0, \frac{3}{5}]$
$1 < \beta$	$(2,0)$	2	$(1,0,0)$	0

Now, the slopes of $V(\beta)$ correspond to $y_2^*(\beta) + y_3^*(\beta)$, since in this case the third primal constraint never binds, so $y_3^*(\beta) = 0$ for all β.