Economics 4113: Midterm 2—Suggested Answers

1. We are given the problem

$$\max_{x,y \geq \mathbf{0}} \alpha \ln(x_1) + \beta \ln(y_1) + \delta(\alpha \ln(x_2) + \beta \ln(y_2)) \text{ subject to}$$
$$p_1 x_1 + q_1 y_1 \leq I_1 \quad \text{and} \quad p_2 x_2 + q_2 y_2 \leq I_2.$$

To solve it, take the Lagrangean:

$$\mathcal{L} = \alpha \ln(x_1) + \beta \ln(y_1) + \delta(\alpha \ln(x_2) + \beta \ln(y_2)) + \lambda_1(I_1 - p_1 x_1 - q_1 y_1) + \lambda_2(I_2 - p_2 x_2 - q_2 y_2)$$

The objective is concave and the constraints are linear, so Kuhn-Tucker conditions are necessary and sufficient. Since utility is strictly increasing, budget constraints will hold with equality, and since marginal utility is unbounded at 0, first-order conditions hold with equality, too. They are for every $i = 1, 2$:

$$\frac{\alpha}{x_i} = \lambda_i p_i \qquad \frac{\beta}{y_i} = \lambda_i q_i.$$

Therefore, $p_i x_i / \alpha = q_i y_i / \beta$ for every i. Substituting into the budget constraints,

$$x_i^* = \frac{\alpha I_i}{p_i} \qquad y_i^* = \frac{\beta I_i}{q_i}.$$

Substituting this into the previous equations, we obtain

$$\lambda_1 = \frac{1}{I_1} \qquad \lambda_2 = \frac{\delta}{I_2}.$$

Finally, plugging in x_i^* and y_i^* into $\mathcal{L}$, we obtain

$$V(p, q, I_1, I_2) = \alpha \ln(\alpha I_1/p_1) + \beta \ln(\beta I_1/q_1) + \delta(\alpha \ln(\alpha I_2/p_2) + \beta \ln(\beta I_2/q_2)) = \ln I_1 + \delta \ln I_2 + C,$$

where C is a constant that does not depend on I_i.

2. $W(s) = \ln(I_1 - s) + \delta \ln(I_2 + (1+r)s) + C$. Its derivatives are

$$W'(s) = \frac{-1}{I_1 - s} + \frac{\delta(1+r)}{I_2 + (1+r)s}$$
$$W''(s) = \frac{-2}{(I_1 - s)^2} - \frac{2\delta(1+r)^2}{(I_2 + (1+r)s)^2} < 0.$$

Since $W''(s) < 0$ for all s, it follows that W is a concave function.

The optimal savings decision can be found by solving for s^* from the condition $W'(s^*) = 0$. This gives

$$s^* = \frac{\delta(1+r)I_1 - I_2}{(1+\delta)(1+r)}.$$

Therefore, $s^* = 0$ exactly if $\delta(1+r)I_1 = I_2$. If $\delta = 0$, the optimal decision is $s^* = -I_2/(1+r)$, i.e., the consumer brings all his money forward to consume only on date 1.

Now suppose that $\delta = 1/(1+r) > 0$. Optimal expenditure becomes

$$\widehat{I}_1 = I_1 - s^* = I_1 - \frac{\delta(I_1 - I_2)}{1+\delta} = \frac{I_1 + \delta I_2}{1+\delta}$$
$$\widehat{I}_2 = I_2 + (1+r)s^* = I_2 + \frac{I_1 - I_2}{1+\delta} = \frac{I_1 + \delta I_2}{1+\delta}.$$

To find the expenditure ratio, notice that by the envelope theorem

$$-\lambda_1^* + \lambda_2^*(1+r) = 0 \qquad \Rightarrow \qquad \widehat{I}_2 = \delta(1+r)\widehat{I}_1.$$

Therefore, $\delta = 1/(1+r)$ implies that $\widehat{I}_1 = \widehat{I}_2$ and the intertemporal expenditure ratio equals one. The intertemporal consumption ratios are given by

$$\frac{x_1^*}{x_2^*} = \frac{\alpha \widehat{I}_1 p_2}{\alpha \widehat{I}_2 p_1} = \frac{p_2}{p_1}$$

and similarly $y_1^*/y_2^* = q_2/q_1$.