Economics 4113, Spring 2010. Instructor: David Rahman, University of Minnesota.
REQUEST: Please substantiate your answers.

Homework 1—Due February 2, 2010

1. Given a matrix $A \in \mathbb{R}^{m\times n}$, define

$$\text{span } A \;=\; \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ s.t. } Ax = y\}.$$

Intuitively, span A is the set of vectors that can be reached by multiplying a vector by the matrix A.

   (a) Prove that the span of a matrix is a linear subspace.

   (b) Sketch the span of the following matrix:

$$A \;=\; \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

   (c) Sketch the span of the following matrix:

$$A \;=\; \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$$

   (d) Sketch the kernel of the matrices in (b) and (c) above. Recall that the kernel of a matrix A is defined as

$$\ker A \;=\; \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}.$$

2. Given a matrix A, the cone generated by A is defined as

$$\text{cone } A \;=\; \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n_+ \text{ s.t. } Ax = y\},$$

where $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) : x_i \geq 0 \; \forall i\}$ is the non-negative orthant of $\mathbb{R}^n$. (Notice how cone A differs from span A.)

   (a) Sketch the cones generated by the matrices defined in problems 1.(b) and 1.(c) above.

   (b) Consider the matrix B that remains from the one in problem 1.(b) after eliminating the second column. What is the cone generated by this matrix? If $b = (1, 0)$, decide whether or not the system $Bx = b$ has a solution $x \geq \mathbf{0}$.

(c) Sketch the cone generated by the following matrix:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 3 & -2 \end{bmatrix}$$

3. Consider the vectors $a = (0, 1, -2)$, $b = (1, 1, 1)$, $c = (1, 2, 3)$ and $d = (2, 0, 3)$. Are these four vectors linearly independent? If so, prove it. If not, exhibit a linear combination of these vectors that yields zero.

4. Let $a_1, \ldots, a_m$ be m-vectors. Prove that the equations

$$a_i y = \beta_i \qquad \forall i \in \{1, \ldots, m\}$$

have a unique solution y if and only if the equations

$$a_i y = 0 \qquad \forall i \in \{1, \ldots, m\}$$

have no nonzero solution. (Hint: It's not as difficult as it looks!)

5. Sketch the hyperplane generated by the following equation in $\mathbb{R}^2$:

$$x_1 - 2x_2 = 3.$$

Identify the set of vectors in $\mathbb{R}^2$ such that $x_1 - 2x_2 < 3$, as well as the set of points such that $x_1 - 2x_2 > 3$. Now repeat this exercise with the equation $x_1 + 2x_2 = 3$ instead.

6. This question is about finding solutions to equations and inequalities.

(a) Find all solutions of the equations

$$\begin{aligned} 2x + 3y - z + w &= 0 \\ x - 5y + 2z &= 0. \end{aligned}$$

(b) Decide whether or not the following equations have a solution:

$$\begin{aligned} 2x + 3y &= 1 \\ x - 3y &= 1 \\ -x + y &= 0. \end{aligned}$$

(c) Decide whether or not the following equations have a non-negative solution:

$$\begin{aligned} x + 3y - 5z &= 2 \\ x - 4y - 7z &= 3. \end{aligned}$$

(d) Decide whether or not the following inequalities have a solution:

$$\begin{aligned} 4x - 5y &\geq 3 \\ -2x - 7y &\geq 1 \\ -2x + y &\geq -2. \end{aligned}$$

(e) Find a solution for the inequalities

$$\begin{aligned} 5x - 4y &\leq 7 \\ -3x + 3y &\leq -5. \end{aligned}$$

Prove that there is no non-negative solution to these inequalities.

(f) Do the following equations have a non-negative solution?

$$\begin{aligned} 3x - 5y + 2z &= 0 \\ 2x - 4y + z &= 0. \end{aligned}$$

7. Show that a set of n homogeneous inequalities in n unknowns always has a nonzero solution.

8. Prove the following statements given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$:

(a) Either $Ax \geq b$ has a non-negative solution x or there is a non-negative solution y such that $yA \leq 0$ and $yb > 0$, but not both.

(b) Either $Ax = \mathbf{0}$, $\sum_i x_i = 1$ has a non-negative solution or there exists $y \in \mathbb{R}^m$ such that $yA \gg \mathbf{0}$, but not both.

(c) Either $Ax = \mathbf{0}$, has a nonzero, non-negative solution or there exists $y \in \mathbb{R}^m$ such that $yA \gg \mathbf{0}$, but not both.

(d) Either $Ax \leq b$ has a solution or there exists $y \in \mathbb{R}^m_+$ such that $yA \geq \mathbf{0}$ and $yb < 0$, but not both.

(e) Either there exists $x \gg \mathbf{0}$ such that $Ax = \mathbf{0}$ or there exists $y \in \mathbb{R}^m$ such that $yA > \mathbf{0}$, but not both.

(f) The system $Ax \ll b$ has a solution if and only if $y = \mathbf{0}$ is the only solution to $\{yA = \mathbf{0}, yb \leq 0, y \geq \mathbf{0}\}$.

(g) Let $F = \{x \in \mathbb{R}^n : Ax \leq \mathbf{0}\}$, $c \in \mathbb{R}^n$ and $G = \{x \in \mathbb{R}^n : cx \leq 0\}$. Prove that $F \subset G$ if and only if there exists $y \in \mathbb{R}^m_+$ such that $c = yA$.