

Homework 2—Due February 18, 2009

1. Consider the following Markov chains.

(a) Let A be the following matrix:

$$A = \begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix}.$$

Describe the set of all invariant distributions of A when: (i) both p and q are strictly positive and strictly less than one, (ii) $p = q = 0$, (iii) $p = q = 1$, (iv) $p = 1$ and $q = 0$, and (v) $p = 0$ and $q = 1$.

(b) Let $0 < \alpha, \beta, \gamma < 1$, $\alpha + \beta = 1$, and

$$A = \begin{bmatrix} 1-\gamma & \alpha\gamma & \beta\gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find A^t for all $t \geq 0$. Describe all the invariant distributions of A . Is there an invariant distribution with positive probability on the first state?

(c) A person's educational attainment level can be divided into three categories: four-year degree or more, high-school diploma, and neither. For the offspring of people with four-year degrees or more, 80 percent also attain a four-year degree, 10 percent attain only a high-school diploma, and 10 percent attain neither. For children of people with high-school diplomas, 60 percent attain a four-year degree, 20 percent a high-school diploma, and 20 percent neither. Finally, for those without even a high-school diploma, 40 percent of children attain neither, 30 percent obtain a high school diploma, and 30 percent attain a four-year degree. Every person has exactly two children with someone of the same educational attainment. Write down the Markov chain that shows the distribution of educational attainment in the population. Can you find an invariant distribution? With what probability does the grandchild of a person with neither a four-year degree nor a high-school diploma have a four-year degree?

- (d) Consider the same data as in question (b) above, except that now a person with a four-year degree only has one child and a person with a high-school diploma has exactly three children. The population begins with no educational attainment whatsoever. What is the fraction of grandchildren with a four-year degree? Does the population grow or shrink?
(Hint: The “frequencies” won’t add up to one!)
2. A *European put* is an option contract that gives its holder the right to sell a stock at a pre-specified date, say tomorrow, at a prearranged price, K . If tomorrow’s stock price S satisfies $S < K$ then the option holder’s optimal strategy is to exercise her put option, i.e., sell the stock at price K to obtain a benefit of $K - S$. Otherwise, if $S \geq K$ then the put option becomes worthless.
- (a) Sketch the graph of the terminal value of a put option with strike price K as a function of the terminal value of the stock S .
- (b) If the stock price increases by a factor of u with probability p and decreases by a factor of d with probability $1 - p$ and interest rates on bonds are equal to r , determine the price of a European put.
- (c) Consider the same data as in part (b) above. An *American put* is an option contract that gives its holder the right to sell a stock *at any time until* a pre-specified date, say tomorrow, at a prearranged price, K . Consider holding an American put option today. Is early exercise ever optimal? If not, prove it. Otherwise, find conditions on the parameters in (b) that make early exercise optimal.
3. Consider the following multi-period version of the previous model. There are three periods, indexed by $t = 0, 1, 2$, with $t = 0$ signifying today. There are two assets: a bond that pays an interest rate equal to r every period, and a stock. In every period, the price of the stock can either increase by a factor of u with probability p or decrease by a factor of d with probability $1 - p$.
- (a) A *European call* with maturity date T and strike price K gives its holder the right to buy a stock at price K on date T . Suppose that $T = 2$. Calculate the price of the European call ...
- i. ... in period $t = 1$ assuming that the stock price went up.
 - ii. ... in period $t = 1$ assuming that the stock price went down.
 - iii. ... in period $t = 0$.

(*Hint*: Start at the end of the event tree and work back node by node.)

- (b) Repeat the previous exercise for an American call with maturity date $T = 2$ and strike price K .

(*Hint*: Derive the holder's optimal exercise strategy at every node.)

- (c) Is it ever optimal to exercise the American call early, i.e., at any period t less than T ? Use your answers in (a) and (b) above.