Homework 2—Due March 11, 2010

1. Derive the dual of the following primal problem:

$$\begin{aligned} V \;=\; \max_{x_1,x_2} \; & x_1 + 2x_2 \quad \text{s.t.} \\ & x_1 + \tfrac{8}{3}x_2 \leq 4, \\ & x_1 + x_2 = 2, \\ & 2x_1 \geq 3, \\ & x_1 \geq 0. \end{aligned}$$

2. Write down the duals of the following LPs: (a) $\max\{cx : Ax = b, x \geq 0\}$, (b) $\min\{cx : Ax = b, x \geq 0\}$, (c) $\max\{cx : Ax \leq b\}$ (d) $\min\{cx : Ax \geq b\}$.

3. Show that the following linear program is feasible but has no optimal solution:

$$\begin{aligned} \max_{x \geq \mathbf{0}} \; & x_1 + x_2 \quad \text{s.t.} \\ -3x_1 + 2x_2 & \leq -1, \\ x_1 - x_2 & \leq 2. \end{aligned}$$

   Write down the dual of this problem. Is this dual feasible?

4. Find the optimal primal and dual solutions to the following LP:

$$\max_{x \geq \mathbf{0}} \; x_1 + x_2 - 3x_3 \quad \text{s.t.} \;\; x_1 + 2x_2 - 3x_3 = 4, \;\; 4x_1 + 5x_2 - 9x_3 = 13.$$

5. Consider the following primal problem:

$$\begin{aligned} \max_{x} \; & x_1 + x_2 + x_3 + x_4 \quad \text{s.t.} \\ & x_1 + x_2 \leq 3, \\ & x_3 + x_4 \leq 1, \\ & x_2 + x_3 \leq 1, \\ & x_1 + x_3 \leq 1. \end{aligned}$$

   Show that $x = (1, 1, 0, 1)$ solves this primal problem by solving the dual.

6. Calculate the functions $V(b)$ and $V'(b)$ (the derivative of V), where

$$\begin{aligned} V(b) &= \max_{x \geq \mathbf{0}} \ x_1 + 2x_2 \ \text{ s.t.} \\ & \qquad x_1 + \tfrac{8}{3}x_2 \leq 4, \\ & \qquad x_1 + x_2 \leq b, \\ & \qquad 2x_1 \leq 3. \end{aligned}$$

How do these compare to the set of optimal dual solutions $y_2^*(b)$?

7. Use strong duality to prove the Theorem of the Alternative.

8. Let $V = \max\{\sum_{j=1}^n c_j x_j : \sum_{j=1}^n a_j x_j \leq b, \ x \geq \mathbf{0}\}$. Assume that all c_j and a_j are positive. Show that $V = b \max_j c_j/a_j$.

9. Convert the following optimization into a linear program.

$$\begin{aligned} \min_{x,y,z} \ |x| + |y| + |z| \quad \text{s.t.} \\ x + y \leq 1, \\ 2x + z = 3. \end{aligned}$$

10. Let F be a function of $x \in \mathbb{R}_+^n$ and $y \in \mathbb{R}_+^m$. The pair (x^*, y^*) is called a *saddle point* of F if

$$F(x^*, y) \ \geq \ F(x^*, y^*) \ \geq \ F(x, y^*) \qquad \forall (x, y) \geq \mathbf{0}.$$

Recall that a dual pair of linear programs in standard form is given by

$$\text{(P)} \quad \max_{x \geq \mathbf{0}} \{cx : Ax \leq b\} \qquad\qquad \text{(D)} \quad \max_{y \geq \mathbf{0}} \{yb : yA \geq c\}$$

(a) Assuming that (x^*, y^*) are optimal solutions for a pair of dual linear programs in standard form, show that (x^*, y^*) is a saddle point of the function

$$F(x, y) \ = \ cx + yb - yAx.$$

(b) Prove that if (x^*, y^*) is a saddle point of the function F from part (a) above then x^* and y^* are optimal solutions for the corresponding dual linear programs in standard form.