

Problem Set 4—Due April 13, 2010

1. Consider the following optimization problem, called the *budget* problem.

$$\max_{x,y \geq 0} \{U(x,y) : px + qy \leq I\},$$

where  $\alpha, \beta, p, q, I > 0$ . Intuitively,  $U(x, y)$  is the utility of a consumer for two goods, where  $x$  denotes the amount consumed of the first good and  $y$  the amount of the second good. The consumer's income is  $I$  dollars, the dollar price of the first good is  $p$  and that of the second good is  $q$ . The constraint  $px + qy = I$  states that the consumer's expenditure on the two goods must equal his income.

- (a) Let  $U(x, y) = \alpha \ln(x) + \beta \ln(y)$ . Write down the Lagrangian and derive necessary conditions for an optimal solution. Calculate the optimal solution  $x^*$  and  $y^*$  as a function of the parameters of the problem. Does the amount demanded for the first good depend on the price of the second good? How do expenditure shares (the expenditure share of a good is the amount of money spent on the good divided by total expenditure, in this case  $I$ ) depend on prices and income? Find the Lagrange multiplier associated with the budget constraint as a function of the parameters of the problem. This is called the “marginal utility of money.”
- (b) Repeat (a) with  $U(x, y) = x^\alpha y^\beta$ . Comment.
- (c) Repeat (a) with  $U(x, y) = \alpha x + \beta y$ , where  $\alpha, \beta > 0$  and  $\alpha + \beta = 1$ . Now suppose that consumption takes time, specifically, it takes  $t$  units of time to consume each unit of  $x$  and  $s$  units of time to consume each unit of  $y$ . The agent is endowed with  $T$  units of time. Solve the following problem:

$$\max_{x,y \geq 0} \{U(x,y) : px + qy \leq I, tx + sy \leq T\},$$

where still  $U(x, y) = \alpha x + \beta y$ , for all relevant values of  $\alpha$  and  $\beta$ .

2. Consider a producer who chooses capital  $K$  and labor  $L$  at prices  $r$  and  $w$ , respectively, to produce a given amount  $Q$  of output, subject to a production technology that relates output to inputs:  $Q = \sqrt{K} + \sqrt{L}$ .

- (a) Write down the producer's cost-minimization problem assuming that he needs to produce exactly  $Q$  units of output. Cost is defined as the sum of his expenditures on capital and labor.
- (b) Find the Lagrange multiplier of the production technology constraint.
- (c) Now suppose that  $p$  is the (dollar) price of output, and suppose that the producer can choose the amount of output. Write down the producer's problem of maximizing profit (i.e., revenue minus cost) subject to the production technology constraint. How much output should the producer produce? Relate this answer to your answer in (b).
3. Consider an economy with 300 units of labor and 400 units of capital. It takes 3 units of labor and 2 units of capital to produce one pound of apples, and it takes 2 units of labor and 3 units of capital to produce one pound of oranges. Social welfare, as a function of the amount of apples and oranges in the economy, is given by  $W(x, y) = \alpha \ln(x) + \beta \ln(y)$ , where  $x$  denotes the pounds of apples,  $y$  the pounds of oranges, and  $\alpha, \beta$  are both nonnegative with  $\alpha + \beta = 1$ . Calculate the efficient (i.e., welfare-maximizing) production of apples and oranges as a function of the parameters of the problem.
4. Consider a consumer with utility function  $U(x_1, x_2, x_3) = \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) + \alpha_3 \ln(x_3)$  over consumption of three goods, denoted 1, 2 and 3. Suppose that  $\alpha_i > 0$  for every good  $i$  and that  $\alpha_1 + \alpha_2 + \alpha_3 = 1$ . The consumer's budget constraint is given by  $p_1x_1 + p_2x_2 + p_3x_3 \leq I$ , with the usual interpretation on the parameters. In addition, the consumer faces a rationing constraint: he is not allowed to buy more than  $k$  units of good 1. Solve the consumer's budget problem including the additional rationing constraint. Under what condition on the various parameters is the rationing constraint binding?
5. There is a fixed amount  $Y$  of wealth to be allocated in a society consisting of two consumers who "envy" each other in the sense that each consumer's payoff depends on others' consumption as well as own consumption. If consumer 1 receives an amount  $Y_1$  of wealth and consumer 2 receives  $Y_2$  then the payoff to consumer 1 is given by  $U_1(Y_1, Y_2) = Y_1 - kY_2^2$ , and the payoff to consumer 2 equals  $U_2(Y_1, Y_2) = Y_2 - kY_1^2$ , where  $k$  is some fixed positive number.
- Write down the problem of maximizing the sum of consumers' payoffs subject to the resource constraint that  $Y_1 + Y_2 \leq Y$ . Show that if  $Y > 1/k$  then the resource constraint will be slack at the optimum. Interpret.