means the square at P_6 goes to M_4, and this overloads the route from P_6 to M_4.

Problems of the above type provide an important subject for future investigation.

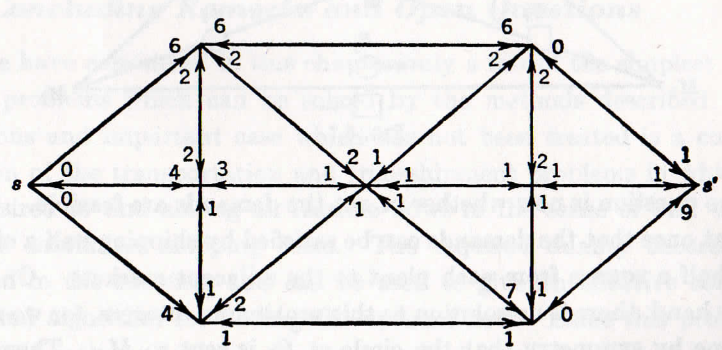
Bibliographical Notes

The max flow min cut theorem which was the starting point for the analysis of this chapter was first proved by Ford and Fulkerson [1]. The feasibility theorem for the simple-assignment problem is due to P. Hall [1]. The second proof of this result is that given by Halmos and Vaughan [1]. The feasibility theorem for the transshipment problem is due to Gale [1]. In the treatment of the optimal-assignment problem we have followed the work of Kuhn [3]. The price equilibrium model is presented here for the first time. The transportation problem is solved in Sec. 7 by the methods of Ford and Fulkerson [2], who are also responsible for the numerical example treated there. The shortest-route problem is described by Ford [1] and the caterer problem was originally proposed by Jacobs [1] and treated as a transportation problem by Prager [1].

Exercises

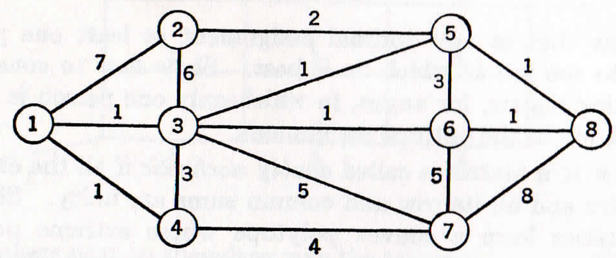
1. Prove that if f is any flow from s to s' in the network N then $f(s, N) = f(N, s')$.

2. Find a maximum flow in the network shown below.



The numbers next to each edge represent the capacities in the directions indicated by the arrows.

3. In the network below, nodes ① and ⑥ are sources with supplies $\sigma(1) = 3$, $\sigma(6) = 5$. Nodes ④ and ⑧ are sinks with demands $\delta(4) = 4$, $\delta(8) = 4$. The numbers on the edges are capacities which are assumed to be the same in both directions. Determine whether this transshipment problem is feasible.



4. Let (S, S') and (R, R') be two minimal cuts in the network N. Show that $(S \cap R, S' \cup R')$ is also a minimal cut. (We remark that this theorem can be stated without defining the notion of a flow, but its proof apparently requires the max flow min cut theorem.)

5. Let Q be the qualification matrix for a simple-assignment problem. A *cover* of Q consists of a set R of rows and C of columns of Q so that every nonzero entry of Q lies in either a row of R or column of C. Prove that the maximum number of people that can be assigned is the minimum number of rows plus columns which will cover Q.

6. Find a minimal cover for the qualification matrix of Sec. 3.

7. Find a solution of the optimal-assignment problem whose rating matrix is the following:

0	15	9	1	3	4	19
2	0	19	11	9	3	14
12	5	17	12	24	15	16
19	11	14	23	16	17	29
20	15	23	22	19	21	24
23	12	16	17	24	25	26
25	16	8	26	21	20	23

8. Let $A = (\alpha_{ij})$ be an $n \times n$ rating matrix, and let $A' = (\alpha_{ij} + \alpha_i)$, where $\alpha_1, \ldots, \alpha_n$ are any n numbers. Show that an assignment is optimal for A' if and only if it is optimal for A.

9. Let $A = (\alpha_{ij})$ be an $n \times n$ rating matrix. Show that there exist numbers α_i such that for the matrix $A' = (\alpha'_{ij}) = (\alpha_{ij} + \alpha_i)$, the value of an optimal assignment is given by the formula

$$\mu = \sum_{j=1}^{n} \max_{i} (\alpha'_{ij})$$

10. Show that in any optimal assignment at least one person is assigned to the job at which he is best. Show how to construct an $n \times n$ rating matrix, for any n, in which only one person is assigned to his best job in the optimal assignment.

11. An $n \times n$ matrix is called *doubly stochastic* if all the entries are nonnegative and all its row and column sums are unity. Show that these matrices form a convex polytope whose extreme points are precisely the *permutation* matrices, i.e., the matrices all of whose row and column vectors are unit vectors. (Hint: Define an equivalent-flow problem and use Theorem 5.4.)

12. For any nonnegative $n \times n$ matrix $A = (\alpha_{ij})$ show that the solutions of the canonical linear program

$$\sum_{i=1}^{n} \lambda_{ij} = 1 \qquad \sum_{j=1}^{n} \lambda_{ij} = 1$$

$$\sum \lambda_{ij}\alpha_{ij} = \text{maximum}$$

form a convex polytope whose extreme points are precisely the optimal assignments.

13. In the problem of equilibrium prices show that the sets of *admissible entries* [satisfying (2) of Sec. 6] correspond to all possible optimal assignments.

14. Solve the 3 by 4 transportation problem whose cost matrix is

4	4	9	3
3	5	8	8
2	6	5	7

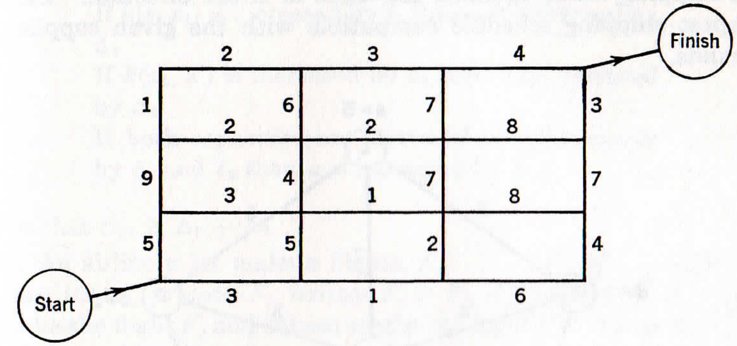
and the supplies are

$$\sigma_1 = 3 \qquad \sigma_2 = 5 \qquad \sigma_3 = 7$$

and the demands are

$$\delta_1 = 2 \qquad \delta_2 = 5 \qquad \delta_3 = 4 \qquad \delta_4 = 4$$

15. Find the shortest path through the following network:



The numbers next to the edges give the mileages.

16. The caterer must supply 60 napkins on Monday, 50 on Tuesday, 80 on Wednesday, 40 on Thursday, 50 on Friday. Napkins cost 5 cents, can be laundered in 1 day for 2 cents and in 2 days for 1 cent. What is the most economical plan for the caterer?

17. The following is a generalization of the transportation problem. In a network N the cost of shipping one unit from x to y is $c(x, y)$. The demand at node x is $d(x)$, and a negative demand is interpreted as a supply. Assuming $d(N) = 0$, show that the problem of satisfying the demand at minimum cost is correctly given as follows: Find a nonnegative function g on edges of N such that

$$\sum_{x,y} g(x, y)c(x, y) \text{ is a minimum} \tag{1}$$

subject to

$$g(N, x) - g(x, N) \geqq d(x) \qquad \text{for all } x \tag{2}$$

Show that the following is the correct dual problem by proving the duality theorem: Find a function π on N such that

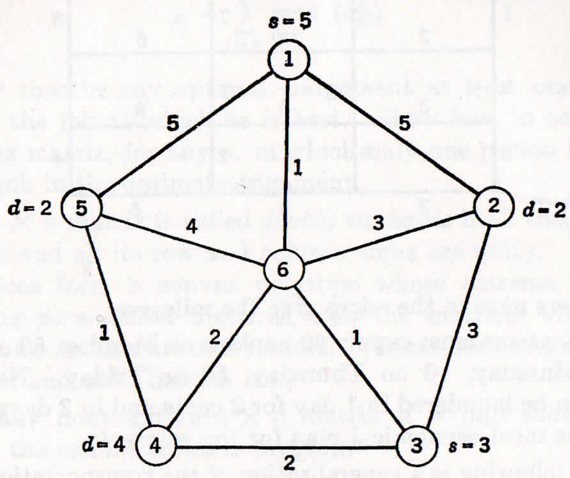
$$\sum_{x} \pi(x)d(x) \text{ is a maximum} \qquad (1)^*$$

subject to

$$\pi(y) - \pi(x) \leqq c(x, y) \qquad \text{for all } x, y \qquad (2)^*$$

18. In the network below, nodes ① and ③ are sources with supplies $s(1) = 5$, $s(3) = 3$. Nodes ②, ④, and ⑤ are sinks with demands

$d(2) = 2$, $d(4) = 4$, $d(5) = 2$. The numbers next to the edges are unit shipping costs, assumed the same in either direction. Find the cheapest shipping schedule compatible with the given supplies and demands.



19. Consider a transportation problem in which there is a capacity κ_{ij} as well as a cost γ_{ij} from the plant P_i to the market M_j. Show that a feasible shipping schedule ξ_{ij} is optimal if and only if there exist prices $\pi_1, \ldots, \pi_m, \pi'_1, \ldots, \pi'_n$ such that

$$\text{if } \pi'_j - \pi_i > \gamma_{ij} \quad \text{then } \xi_{ij} = \kappa_{ij} \tag{1}$$
$$\text{if } \pi'_j - \pi_i < \gamma_{ij} \quad \text{then } \xi_{ij} = 0 \tag{2}$$

(Hint: See Exercise 16 of Chap. 3.)

20. In a network N with source s and sink s' let x_1 and x_2 be any nodes of N. Suppose

If capacity $k(s, x_1)$ is increased by δ_1 then the value μ of the maximum flow is increased by Δ_1 (1)

If $k(s, x_2)$ is increased by δ_2 then μ is increased by Δ_2 (2)

If $k(s, x_1)$ and $k(s, x_2)$ are increased simultaneously by δ_1 and δ_2 then μ is increased by Δ_{12} (3)

Prove that $\Delta_{12} \leqq \Delta_1 + \Delta_2$.

21. As in Exercise 20 suppose

If $k(s, x_1)$ is increased by δ_1 then μ is increased by Δ_1 (1)

If $k(x_2, s')$ is increased by δ_2 then μ is increased by Δ_2 (2)

If both capacities are increased simultaneously by δ_1 and δ_2 then μ is increased by Δ_{12} (3)

Prove that $\Delta_{12} \geqq \Delta_1 + \Delta_2$.

22. An airline must make n flights, $F_1, \ldots, F_n$, in a certain year. We say that F_i *precedes* F_j, written $F_i \succ F_j$, if it is possible for a plane to make the flight F_i and subsequently the flight F_j. From the meaning of the problem it is clear that the relation $\succ$ satisfies

$$\text{if } F_i \succ F_j \text{ and } F_j \succ F_k, \text{ then } F_i \succ F_k \tag{1}$$
$$\text{if } F_i \succ F_j, \text{ then } F_j \not\succ F_i \tag{2}$$

Two flights are called *noncomparable* if neither precedes the other. Prove: *the minimum number of planes needed to meet a given schedule is equal to the maximum number of mutually noncomparable flights.*

Hint: Consider the $n \times n$ simple-assignment problem with qualification matrix $A = (\alpha_{ij})$, where $\alpha_{ij} = 1$ if and only if $F_i \succ F_j$. Suppose the maximum number of assignments and therefore the minimum cover of A is k (see Exercise 5).

a. Using the k assignments, show that all flights can be made by $n - k$ planes.

b. Show that if a cover of A includes both the ith row and the ith column of A it is not minimal. Hence show that the flights F_i for which neither the ith row nor column lies in the minimal cover form a set of $n - k$ noncomparable flights.