Abstract Single Crossing and the Value Dimension

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Abstract

When auctioning an indivisible good without consumption externalities, abstract single crossing is necessary and sufficient to implement an efficient allocation in ex post equilibrium.

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1 Introduction

This short note characterizes deterministic, ex post incentive compatible (EPIC) allocations as those that are *weakly monotone* in an environment where a single, indivisible good must be allocated. Consumption externalities are not allowed, but arbitrarily interdependent values are. Moreover, the characterization applies to any type space, not necessarily convex or ordered. An immediate corollary of this result is an *abstract single crossing* condition (ASC) on individual utility functions that is both necessary and sufficient for the EPIC implementation of an efficient allocation.

Weak monotonicity and ASC both point to the dimension of value as the relevant one. Intuitively, weak monotonicity reads as follows. Fix an individual, say i, a profile of types for others, and consider any two types for i. If individual i is allocated the good at his first type and values it more at his second type then he should also be allocated the good at his second type. Similarly, ASC requires that if individual i values it more than others at his first type and he values it more at his second type than it his first, then he should value it more than others at his second type, too.

Weak monotonicity and ASC isolate the determinants of EPIC and efficient EPIC allocations, respectively, regardless of whether or not type spaces are convex, ordered, utility functions are monotone and differentiable, in contrast with the literature.

Maskin (1992), to whom the notion of single crossing in auctions is attributable, noted that multidimensional signals may limit the efficiency properties of auctions. Since then, an extensive literature on the single crossing condition in auctions has emerged, mostly within the standard framework (ordered, convex type spaces with monotone, differentiable utility functions). As such, most results rely on this framework, and yield conditions that to some readers appear technical, unintuitive, or specific.

An exception to this trend is a recent paper by Saks and Yu (2005), whose argument is considerably streamlined by Vohra (2007). Building on Bikhchandani et al. (2006), they prove that weak monotonicity is necessary and sufficient for an allocation to be EPIC on convex domains in otherwise arbitrary environments. Their result may be viewed as a refinement of one by Rochet (1987) characterizing EPIC allocations as cyclically monotone on *arbitrary* type spaces. A technical contribution of this note is to show that when auctioning off a single indivisible good without consumption externalities, convexity may be dropped. Another contribution is to steer away from multidimensionality per se and towards conditions that only involve utility rankings.

2 Model

Consider a population of finitely many individuals, collected in the set $I = \{1, \ldots, n\}$. Each individual i has private information represented by a nonempty type space T_i with typical element t_i. The product space of type profiles is given by

$$T = \prod_{i=1}^{n} T_i,$$

with typical element $t = (t_1, \ldots, t_n)$.

A single, indivisible good is to be allocated. For each i, let

$$v_i : T \to [0, \infty)$$

denote Mr. i's utility from the good, and let $\mathbf{v} = (v_1, \ldots, v_n)$ be the profile of utility functions. If Mr. i does not obtain the good then he draws zero utility.

A deterministic *allocation* is any map $\mathbf{z} : T \to I \cup \{0\}$, where $\mathbf{z}(t)$ stands for the individual to whom the good is allocated when t is profile of types, and $\mathbf{z}(t) = 0$ means that nobody gets it.

Individuals have quasi-linear preferences over money payments. A *payment scheme* is any function $\xi : I \times T \to \mathbb{R}$ that maps profiles of reported types to individual money payments. Thus, $\xi_i(t)$ denotes the money paid by Mr. i if t is the profile of types.

Definition 1. An allocation $\mathbf{z}$ is *EPIC* if there is a payment scheme ξ such that

$$\forall i \in I, t \in T, s_i \in T_i, \qquad v_i(t, \mathbf{z}(s_i, t_{-i})) - v_i(t, \mathbf{z}(t)) \;\leq\; \xi_i(s_i, t_{-i}) - \xi_i(t),$$

where, given s and t, $v_i(t, \mathbf{z}(s)) := v_i(t)\mathbf{1}_{\{\mathbf{z}=i\}}(s)$ and $\mathbf{1}_{\{\mathbf{z}=i\}}$ is the indicator function for whether or not i gets the good (so $\mathbf{1}_{\{\mathbf{z}=i\}}(s) = 1$ if $\mathbf{z}(s) = i$ and zero otherwise).

The left-hand side represents Mr. i's expected utility gain from mis-reporting assuming observation of t, whereas the right-hand side captures Mr. i's expected contractual loss associated therewith. Hence, the inequalities require that every individual's utility gain from mis-reporting must be outweighed by its associated contractual loss.

Definition 2. An allocation $\mathbf{z}$ is *weakly monotone* if

$$\forall i \in I, t \in T, s_i \in T_i, \qquad v_i(t) < v_i(s_i, t_{-i}) \quad \& \quad \mathbf{z}(t) = i \quad \Rightarrow \quad \mathbf{z}(s_i, t_{-i}) = i.$$

Proposition 1. *An allocation* $\mathbf{z}$ *is EPIC if and only if it is weakly monotone.*

Proof. According to Rochet (1987), an allocation is EPIC if and only if it is cyclically monotone. But cyclic monotonicity implies weak monotonicity, yielding sufficiency. For necessity, fix any individual i, profile of others' types t_{-i}, and any finite cycle $(t_i^0, \ldots, t_i^m)$ of types for i. Let $(z_0, ..., z_m)$ be the induced cycle of social choices, i.e., $z_k = \mathbf{z}(t_i^k, t_{-i})$, and $(v_i^0, \ldots, v_i^m)$ the induced cycle of values, i.e., $v_i^k = v_i(t_i^k, t_{-i})$. Let $x_k = 1$ if player i gets the good, i.e., $z_k = i$, and 0 otherwise. Define the index sets

$$M = \{k : x_k = 0 \ \& \ x_{k+1} = 1\} \qquad \text{and} \qquad N = \{\ell : x_\ell = 1 \ \& \ x_{\ell+1} = 0\},$$

where $m + 1 = 0$. Clearly, $|M| = |N|$. For cyclic monotonicity, we require that

$$\begin{aligned} 0 \ &\leq \ \sum_{k=0}^{m} v_i(t_i^k, t_{-i}, z_k) - v_i(t_i^{k+1}, t_{-i}, z_k) \\ &= \ \sum_{k=0}^{m} (v_k - v_{k+1}) x_k \ = \ \sum_{k=0}^{m} v_{k+1}(x_{k+1} - x_k) \\ &= \ \sum_{k \in M} v_{k+1}(x_{k+1} - x_k) + \sum_{\ell \in N} v_{\ell+1}(x_{\ell+1} - x_\ell) \\ &= \ \sum_{k \in M} v_{k+1} x_{k+1} - \sum_{\ell \in N} v_{\ell+1} x_\ell \ = \ \sum_{k \in M} v_{k+1} - \sum_{\ell \in N} v_{\ell+1}. \end{aligned}$$

By weak monotonicity, $v_{\ell+1} \leq v_{k+1}$ for every $k \in M$ and $\ell \in N$. Since M has the same cardinality as N, it follows that

$$\sum_{k \in M} v_{k+1} \ \geq \ \sum_{\ell \in N} v_{\ell+1}.$$

Therefore, $\mathbf{z}$ is cyclically monotone, hence EPIC, too. □

This result holds for multi-dimensional types. It follows that any weakly monotone allocation is EPIC. In fact, if we focus on *efficient* allocations, it redefines single-crossing in a way that is independent of the type space's dimension.

Definition 3. The profile $\mathbf{v} = (v_1, \ldots, v_n)$ exhibits *abstract single crossing* (ASC) if

$$\begin{aligned} \forall (i, t, s_i), \qquad & v_i(t) < v_i(s_i, t_{-i}) \quad \& \quad \forall j \neq i, \ v_i(t) \geq v_j(t) \\ & \qquad \Rightarrow \quad \forall j \neq i, \ v_i(s_i, t_{-i}) \geq v_j(s_i, t_{-i}). \end{aligned}$$

An allocation $\mathbf{z}^*$ is *efficient* if

$$\forall t \in T, \qquad \mathbf{z}^*(t) \in \arg\max_{i \in I} v_i(t).$$

By Proposition 1, weak monotonicity implies the following.

Corollary 1. *An efficient allocation is EPIC if and only if the profile* $\mathbf{v} = (v_1, \ldots, v_n)$ *of utility functions exhibits abstract single crossing.*

ASC says that if it is efficient to give the good to Mr. i then it should also be efficient to give it to him when his type yields a higher payoff and everyone else's type is fixed. Again, abstract single crossing is independent of the dimensionality of the type space.

3 Examples

Consider the following "obvious" counterexample. In a single good environment with two individuals, suppose that $v_i(t_i, t_j) = t_i + 2t_j$, where $t_i \in [0, 1]$. Then the efficient allocation fails to satisfy weak monotonicity. However, it is easy to check that lots of others do. For instance, the allocation given by $\mathbf{z}(t) = j(t)$ where $j(t) \in \arg\max_i t_i$ is EPIC but inefficient. Also, it is easy to see that abstract single crossing fails in Example 4.2 (with multidimensional type spaces) of Jehiel and Moldavanu (2001).

To see how abstract single crossing applies to environments with multi-dimensional type spaces, suppose that there are two individuals, with respective type spaces given by $T_1 = [0, 1]$ and $T_2 = [0, 1] \times [0, 1]$, with typical elements $t_1 \in T_1$ and $(t_2, s_2) \in T_2$. Utility functions are given by $v_1(t_1, t_2, s_2) = t_1 + s_2$ and $v_2(t_1, t_2, s_2) = t_2 + s_2$. Mr. 1's type describes his private value, t_1, for the good, whereas Mr. 2's type describes both his private value, t_2, and a common value component, s_2. Ideally, Mr. 1 should not get the good if and only if $t_1 < t_2$. However, such allocation is not weakly monotone. In fact, by Proposition 1 every EPIC allocation must be "adapted" to each individual's iso-value lines, which may be thought of as "bids" in its indirect implementation.

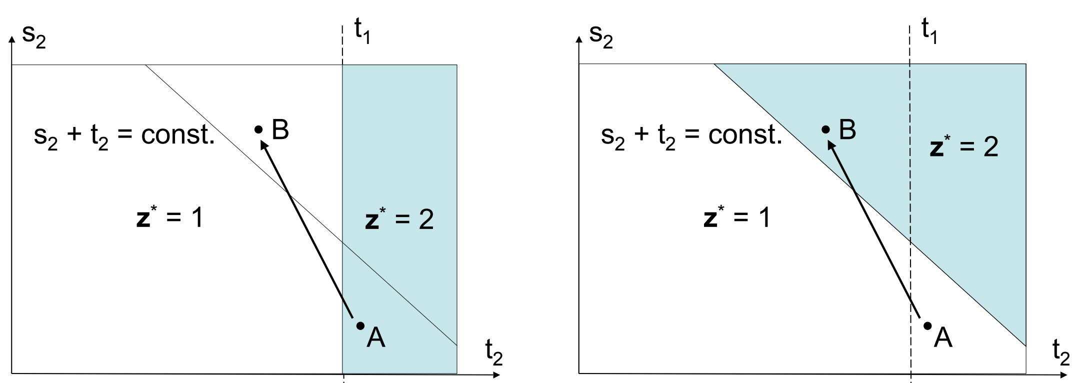

Figure 1: Failure of abstract single crossing (left) and weak monotonicity (right).

This can easily be seen from Figure 1 below. It depicts Mr. 2's type space at some realization of Mr. 1'a type, say t_1. The efficient allocation should give the good to Mr. 2 when his type is given by point A, and to Mr. 1 when it is given by point B. However, it is clear that such an allocation must weak monotonicity, since going from A to B involves crossing over iso-value lines.

For another example, consider the same environment except for Mr. 2's utility function, which becomes $v_2(t_1, t_2, s_2) = t_2 + |s_2 - .5|$. Now the ordering of types is inconsistent between both individuals: v_1 is monotone in the common-value component, s_2, whereas v_2 is not. Still, weak monotonicity characterizes EPIC allocations to be ones adapted to individual iso-value lines. This can be seen in Figure 2 below.

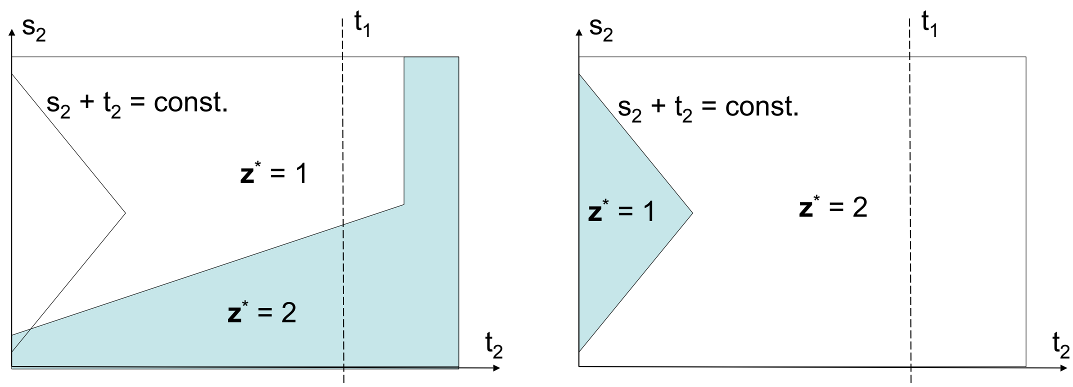

Figure 2: Failure of ASC (left) and weak monotonicity (right) without order.

Of course, insights from the examples above apply also to their "discretized" versions. Therefore, the intuition above applies even when the type space is not convex.

4 Conclusion

EPIC allocations of a single, indivisible good have been characterized in this note by weak monotonicity in the spirit of Bikhchandani et al. (2006), and as a corollary, efficient EPIC ones, too, in environments without consumption externalities but arbitrarily interdependent values. These results are simply derived without reference to any structural properties of the environment except for the value dimension. This suggests that multi-dimensional types and other similar restrictions are not essential to understanding mechanism design. This interpretation may also be drawn from Saks and Yu (2005) in their more general environment save for a convex type space.

Saks and Yu's assume convexity to apply the separating hyperplane theorem. This note is restricted to utility functions that are only one-dimensional (the value of the good being the dimension), so a hyperplane is just a point and convexity of sets becomes unnecessary to separate them by a point.

Apart from its technical contribution, this note makes the following basic argument: Economic conditions to describe efficiency, incentive compatibility, etc., should not be marred by mathematical complexity. A fruitful area for research seems to be to give center stage to this doctrine in the context of auction theory and mechanism design. This includes restricting attention away from multi-dimensional types per se, monotone/differentiable utility functions, etc., despite historical precedent.

A Cyclic Monotonicity

Theorem 1 (Rochet)**.** *An allocation* $\mathbf{z}$ *is EPIC if and only if it is cyclically monotone, i.e., for every* $i \in I$, $t_{-i} \in T_{-i}$ *and finite cycle* $(t_i^0, t_i^1, \ldots, t_i^{m+1})$ *with* $t_i^0 = t_i^{m+1}$,

$$\sum_{k=0}^{m}[v_i(t_i^k, t_{-i}, \mathbf{z}(t_i^k, t_{-i})) - v_i(t_i^{k+1}, t_{-i}, \mathbf{z}(t_i^k, t_{-i}))] \;\geq\; 0. \qquad (***)$$

Proof. For necessity, suppose $\mathbf{z}$ is ex post implementable with transfers ξ and for any (i, t_{-i}), let $(t_i^0, t_i^1, \ldots, t_i^{m+1})$ with $t_i^0 = t_i^{m+1}$ be a finite cycle. For all $k \in \{0, 1, \ldots, m\}$,

$$v_i(t_i^{k+1}, t_{-i}, \mathbf{z}(t_i^k, t_{-i})) - v_i(t_i^{k+1}, t_{-i}, \mathbf{z}(t_i^{k+1}, t_{-i})) \;\leq\; \xi_i(t_i^k, t_{-i}) - \xi_i(t_i^{k+1}, t_{-i}).$$

Adding with respect to k, the right-hand side equals 0, and cyclic monotonicity $(***)$ follows. For sufficiency, fix $t_i^0 \in T_i$ and for any $t \in T$, let

$$W_i(t) \;:=\; \sup \sum_{k=0}^{m}[v_i(t_i^{k+1}, t_{-i}, \mathbf{z}(t_i^k, t_{-i})) - v_i(t_i^k, t_{-i}, \mathbf{z}(t_i^k, t_{-i}))],$$

where the supremum is taken with respect to all finite sequences $(t_i^0, t_i^1, \ldots, t_i^{m+1})$ with $t_i^{m+1} = t_i$. By $(***)$, $W_i(t_i^0, t_{-i}) = 0$. For any $t_i \in T_i$,

$$W_i(t_i^0, t_{-i}) \;\geq\; W_i(t) + v_i(t_i^0, t_{-i}, \mathbf{z}(t)) - v_i(t, \mathbf{z}(t)),$$

which implies that $W_i(t)$ is finite. By definition of $W_i(t)$, for any $s_i \in T_i$,

$$W_i(t) \;\geq\; W_i(s_i, t_{-i}) + v_i(t, \mathbf{z}(s_i, t_{-i})) - v_i(s_i, t_{-i}, \mathbf{z}(s_i, t_{-i})).$$

Finally, substituting $W_i(t) = v_i(t, \mathbf{z}(t)) - \xi_i(t)$ yields ex post incentive compatibility. □

References

Bikhchandani, S., S. Chatterji, A. Sen, R. Lavi, A. Mu'alem, and N. Nisan (2006): "Weak Monotonicity Characterizes Deterministic Dominant-Strategy Implementation," *Econometrica*, 74, 1109–1132. 1, 5

Jehiel, P. and B. Moldavanu (2001): "Efficient Design with Interdependent Valuations," *Econometrica*, 69, 1237–1259. 4

Maskin, E. (1992): *Auctions and Privatization*, Tubingen Mohr (Siebeck), 115–136. 1

Rochet, J. C. (1987): "A Necessary and Sufficient Condition for Rationalizability in a Quasi-Linear Context," *Journal of Mathematical Economics*, 16, 191–200. 1, 3, 6

Saks, M. and L. Yu (2005): "Weak Monotonicity Suffices for Truthfulness on Convex Domains," *Proceedings of the 6th ACM Conference on Electronic Commerce*, EC05, 286–293. 1, 5, 6

Vohra, R. V. (2007): "Paths, Cycles and Mechanism Design," Working Paper. 1