Preliminary Exams

Microeconomic Theory

September 8, 2009

	QI.1	QI.2	QII.1	QII.2	QIII.1	QIII.2	QIV.1	QIV.2
Spring 2009	Pareto optimality	Nonconvexities	Supermodularity	Risk aversion	Linear games			
Fall 2008	Theorem of Pratt	WAPM	Second welfare theorem	Pareto efficiency	Correlated equilibria			
Spring 2008	Risk aversion	GWARP	Core	First welfare theorem		Never a best response		
Fall 2007		Risk aversion	First welfare theorem	Sonnenschein conjecture	Nash equilibrium	Correlated equilibria		
Spring 2007		Risk aversion	Generic approach	First welfare theorem	Extensive form games			
Fall 2006	Profit maximization				Perfect equilibria			
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Fall 2005	Risk aversion		Nonconvexities		Nash equilibrium			
Spring 2005								

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Final 2008/2009	Supermodularity			

Jan Werner

Risk aversion
Exercises II Q1 F2008

Let v_1 and v_2 be two twice-differentiable von Neumann-Morgenstern utility functions. Show that v_1 is more risk averse than v_2 (i.e. $A_1(x) > A_2(x)\ \forall x$) if and only if, for every deterministic $w \in \mathbb{R}$ and every random variable $\tilde{z}$,

$$E[v_2(w+\tilde{z})] \leq v_2(w) \quad \text{implies} \quad E[v_1(w+\tilde{z})] \leq v_1(w)$$

That is, if agent 2 rejects gamble $\tilde{z}$, then so does agent 1.

Solution.

Proof.

- $\Rightarrow$ Let $A_1(x) > A_2(x)$, for all x. Then, by Pratt's theorem $v_1(\cdot) = f(v_2(\cdot))$ where f is a strictly increasing and concave function, and if $E[v_2(w+\tilde{z})] \leq v_2(w)$ we have

$$E[v_1(w+\tilde{z})] = E[f(v_2(w+\tilde{z}))] \leq f(E[v_2(w+\tilde{z})]) \leq f(v_2(w)) = v_1(w)$$

  where the first inequality is the result of Jensen's inequality.

- $\Leftarrow$ Let $E[v_2(w+\tilde{z})] \leq v_2(w)$ imply $E[v_1(w+\tilde{z})] \leq v_1(w)$ for all $w \in \mathbb{R}$, and every random variable $\tilde{z}$. For a random variable $\tilde{z}$ for which $E(\tilde{z}) = 0$ we have by the definition of risk compensation $E[v_2(w+\tilde{z})] = v_2(w - \rho_2(w,\tilde{z}))$. If $E[v_2(w+\tilde{z})] \leq v_2(w)$ then $\rho_2(w,\tilde{z}) \geq 0$, and by assumption this always implies $E[v_1(w+\tilde{z})] \leq v_1(w)$ or $\rho_1(w,\tilde{z}) \geq 0$. Thus $\rho_1(w,\tilde{z}) \geq \rho_2(w,\tilde{z}) \geq 0$. By Pratt's theorem $\rho_1(w,\tilde{z}) \geq \rho_2(w,\tilde{z}) \Leftrightarrow A_1(x) \geq A_2(x)$ for all x. □

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Risk aversion
Exercises II Q2 F2008

Consider an agent with expected utility function and suppose that von Neumann-Morgenstern utility function v is

$$v(y) = -e^{-\alpha x}$$

for some parameter $\alpha > 0$.

(i) Show that risk compensation $\rho(w,\tilde{z})$ is independent of w, for every $\tilde{z}$.

(ii) Show that risk compensation $\rho(w,\tilde{z})$ is an increasing function of parameter α.

You may use the Theorem of Pratt. However, if you use a corollary, you need to show how it follows from the Theorem.

Solution.

Proof.

(i) By definition $E[v(w+\tilde{z})] = v(w - \rho(w,\tilde{z}))$. For the given utility function

$$E[-e^{-\alpha(w+\tilde{z})}] = -e^{-\alpha(w-\rho(w,\tilde{z}))} \Leftrightarrow e^{-\alpha w}E[e^{\alpha\tilde{z}}] = e^{-\alpha w}e^{\alpha\rho(w,\tilde{z})} \Leftrightarrow \rho(w,\tilde{z}) = \alpha^{-1}\ln E[e^{\alpha\tilde{z}}]$$

(ii) We have

$$v_2(x) = -e^{-\alpha_2 x} = -(e^{-\alpha_1 x})^{\frac{\alpha_2}{\alpha_1}} = f(-e^{-\alpha_1 x}) = f(v_1(x))$$

where we define $f(z) = -((-z)^{\frac{\alpha_2}{\alpha_1}})$. If $\alpha_1 < \alpha_2$ function f is a strictly increasing and strictly concave. By Pratt's theorem then $\rho_1(w,\tilde{z}) < \rho_2(w,\tilde{z})$. Alternatively,

$$A(x) = -\frac{v''(x)}{v'(x)} = -\frac{-\alpha^2 e^{-\alpha x}}{\alpha e^{-\alpha x}} = \alpha$$

If $\alpha_2 > \alpha_1$ then $A_2(x) > A_1(x)$, and so by Pratt's theorem $\rho_1(w,\tilde{z}) < \rho_2(w,\tilde{z})$. □

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Stochastic dominance

Exercises II Q3 F2008

Consider two real-valued random variables Y and Z on a probability space. You may think about Y and Z as two contingent claims on a state space. You may assume that the state space is finite. Suppose that Y can take only one of two possible values y_1, y_2 with respective probabilities $\pi_1 > 0$ and $\pi_2 > 0$ such that $\pi_1 + \pi_2 = 1$. Suppose that the expectations of Z conditional on $\{Y = y_1\}$ and $\{Y = y_2\}$ are zero, that is $E(Z|Y = y_1) = E(Z|Y = y_2) = 0$. Prove that $Y + Z$ is more risky than Y.

Solution.

Recall that for two random variables X_1, X_2 on the same probability space, X_2 is said to be more risky than X_1 if $E(X_1) = E(X_2)$ and X_1 second order stochastically dominates X_2; and that X_1 second order stochastically dominates X_2 if and only if for any nondecreasing, concave, continuous function g we have $E(g(X_1)) \geq E(g(X_2))$.

Proof. Suppose that Z takes values $z_1, \ldots, z_n$

i.

$$\begin{aligned}
E(Y+Z) &= \sum_{i=1}^{n}(y_1 + z_i)P(Z = z_i|Y = y_1)\pi_1 + \sum_{i=1}^{n}(y_2 + z_i)P(Z = z_i|Y = y_2)\pi_2 \\
&= \pi_1\left(y_1\sum_{i=1}^{n}P(Z = z_i|Y = y_1) + \sum_{i=1}^{n}z_iP(Z = z_i|Y = y_1)\right) \\
&\quad + \pi_2\left(y_2\sum_{i=1}^{n}P(Z = z_i|Y = y_2) + \sum_{i=1}^{n}z_iP(Z = z_i|Y = y_2)\right) \\
&= \pi_1(y_1 + E(Z|Y = y_1)) + \pi_2(y_2 + E(Z|Y = y_2)) \\
&= \pi_1 y_1 + \pi_2 y_2 = E(Y)
\end{aligned}$$

ii.

$$\begin{aligned} E(g(Y+Z)) &= \sum_{i=1}^{n} g(y_1+z_i)P(Z=z_i|Y=y_1)\pi_1 + \sum_{i=1}^{n} g(y_2+z_i)P(Z=z_i|Y=y_2)\pi_2 \\ &= \pi_1 E(g(Y+Z)|Y=y_1) + \pi_2 E(g(Y+Z)|Y=y_2) \\ &\le \pi_1 g(E(Y+Z|Y=y_1)) + \pi_2 g(E(Y+Z|Y=y_2)) \\ &= \pi_1 g(y_1) + \pi_2 g(y_2) \\ &= E(g(Y)) \end{aligned}$$

□

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Coefficient of absolute risk aversion

Exercises II Q4 F2008

Consider expected utility $\pi v(c_1) + (1-\pi)v(c_2)$ defined on $(c_1, c_2) \in \mathbb{R}^2$ with $\pi \in (0,1)$ and v twice differentiable. Show that the slope of the indifference curve at $c_1 = c_2$ equals $-\frac{\pi}{1-\pi}$ for each $c_1 > 0$. Further, show that the second order derivative (curvature of the indifference curve at $c_1 = c_2$ is proportional to $-\frac{v''(c_1)}{v'(c_1)}$).

Solution.

The indifference curve is characterized as $\pi v(c_1) + (1-\pi)v(c_2) = \overline{u}$. Let $F(c_1, c_2) = \pi v(c_1) + (1-\pi)v(c_2) - \overline{u}$, then the indifference curve $F(c_1, c_2) = 0$ defines an implicit function $c_2(c_1)$. The slope of the indifference curve is

$$\frac{dc_2}{dc_1} = -\frac{\frac{\partial F}{\partial c_1}}{\frac{\partial F}{\partial c_2}} = -\frac{\pi v'(c_1)}{(1-\pi)v'(c_2)}$$

and at $c_1 = c_2$ this becomes $\frac{dc_2}{dc_1} = -\frac{\pi}{(1-\pi)}$. The curvature is

$$\frac{d^2 c_2}{dc_1^2} = \frac{d}{dc_1}\left(\frac{dc_2}{dc_1}\right) = -\frac{\pi v''(c_1)(1-\pi)v'(c_2) - \pi v'(c_1)(1-\pi)v''(c_2)(-1)\frac{\pi v'(c_1)}{(1-\pi)v'(c_2)}}{(1-\pi)^2(v'(c_2))^2}$$

and at $c_1 = c_2$ this becomes

$$\frac{d^2 c_2}{dc_1^2} = -\frac{\pi v''(c_1) + \pi v''(c_2)\frac{\pi}{(1-\pi)}}{(1-\pi)v'(c_2)} = -\frac{\pi}{(1-\pi)^2}\frac{v''(c_1)}{v'(c_1)}$$

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Stochastic dominance

Exercises II Q5 F2008

Give an example of two random variables $\tilde{y}$ and $\tilde{z}$ with $E(\tilde{y}) = E(\tilde{z})$ such that neither $\tilde{y}$ is more risky than $\tilde{z}$ nor $\tilde{z}$ is more risky than $\tilde{y}$.

Solution.

Let $\tilde{y}$ take values $(0, 3)$ with probabilities $(\frac{1}{3}, \frac{2}{3})$ and $\tilde{z}$ take values $(1, 4)$ with probabilities $(\frac{2}{3}, \frac{1}{3})$. Then $E(\tilde{y}) = E(\tilde{z}) = 2$, but

$$\int_{-\infty}^{x} F_{\tilde{y}}(t)dt > \int_{-\infty}^{x} F_{\tilde{z}}(t)dt \qquad \text{for } x \in (0, 2)$$

and

$$\int_{-\infty}^{x} F_{\tilde{y}}(t)dt < \int_{-\infty}^{x} F_{\tilde{z}}(t)dt \qquad \text{for } x \in (2, 4)$$

Thus neither $\tilde{y}$ second order stochastically dominates $\tilde{z}$, nor $\tilde{z}$ second order stochastically dominates $\tilde{y}$.

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Risk aversion

Exercises II Q6 F2008

Give an example of two von Neumann-Morgenstern utility functions v_1 and v_2 such that neither v_1 is more risk averse then v_2 nor v_2 is more risk averse than v_1.

Solution.

Let $v_1(x) = -x^{-1}$ and $v_2(x) = -e^{-x}$. Then

$$A_1(x) = -\frac{v_1''(x)}{v_1'(x)} = -\frac{-2x^{-3}}{x^{-2}} = \frac{2}{x}$$
$$A_2(x) = -\frac{v_2''(x)}{v_2'(x)} = -\frac{-e^{-x}}{e^{-x}} = 1$$

Thus $A_1(x) > A_2(x)$ for $x < 2$, and $A_2(x) > A_1(x)$ for $x > 2$.

In general, for a CRRA utility function $u_1(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ and a CARA utility function $u_2(c) = -e^{-\alpha x}$ we have

$$A_1(x) = -\frac{u_1''(x)}{u_1'(x)} = -\frac{-\sigma x^{-\sigma-1}}{x^{-\sigma}} = \frac{\sigma}{x}$$
$$A_2(x) = -\frac{u_2''(x)}{u_2'(x)} = -\frac{\alpha^2 e^{-\alpha x}}{-\alpha e^{-\alpha x}} = \alpha$$

Thus $A_1(x) > A_2(x)$ for $x < \sigma/\alpha$, and $A_2(x) > A_1(x)$ for $x > \sigma/\alpha$.

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Risk aversion

Exercises II Q7 F2008

Consider an agent whose preferences over risky consumption plans have an expected utility representation with strictly increasing and continuous utility function $v : \mathbb{R} \to \mathbb{R}$. Prove that the agent is risk averse if and only if $Ev(\tilde{z}) \geq Ev(\tilde{y})$ for every $\tilde{y}$ and $\tilde{z}$ such that $E(\tilde{y}) = E(\tilde{z})$ and $\tilde{y}$ is more risky than $\tilde{z}$.

Solution.

Proof.

- $\Rightarrow$ Consider a risk averse agent. By definition this means that $E(v(c)) \leq v(E(c))$ for any risky consumption plan c, and so shkler utility function is concave. If $\tilde{y}$ is more risky than $\tilde{z}$, we know that $E(\tilde{y}) = E(\tilde{z})$ and that $\tilde{z}$ second order stochastically dominates $\tilde{y}$. This in turn means that for any continuous, nondecreasing, concave function g it hold $Eg(\tilde{z}) \geq Eg(\tilde{y})$. Thus this holds also for v.

- $\Leftarrow$ Consider a risky consumption plan $\tilde{y}$, and let $\tilde{z} \equiv E(\tilde{y})$. Then by construction $\tilde{y}$ is more risky than $\tilde{z}$. Using the assumption $Ev(\tilde{y}) \leq Ev(\tilde{z}) = Ev(E(\tilde{y})) = v(E(\tilde{y}))$. Thus the agent is risk averse. □

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Risk aversion

Exercises II Q8 F2008

Let $\tilde{y}$ and $\tilde{z}$ be two normally distributed random variables with the same expected value and with variances σ_y^2 and σ_z^2. Show that $\tilde{y}$ is more risky than $\tilde{z}$ if and only if $\sigma_y^2 \geq \sigma_z^2$.

Solution.

Proof.

- $\Rightarrow$ If $\tilde{y}$ is more risky than $\tilde{z}$, for any nondecreasing concave continuous function g it holds $E(g(\tilde{y})) \leq E(g(\tilde{z}))$. Let $g(x) = -(\tilde{x} - E(\tilde{x}))^2$. Then $-\sigma_y^2 = E(g(\tilde{y})) \leq E(g(\tilde{z})) = -\sigma_z^2$.

- $\Leftarrow$ Assume that $\sigma_y^2 \geq \sigma_z^2$, and that $\tilde{y}$ and $\tilde{z}$ are normally distributed random variables with same expected value. We want to show that for any $x \in \mathbb{R}$ it holds $\int_{-\infty}^{x} F_{\tilde{y}}(t)dt \geq \int_{-\infty}^{x} F_{\tilde{z}}(t)dt$.

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Supermodularity

Final Q1 F2008

Consider a firm with production function $f : \mathbb{R}_+^n \to \mathbb{R}_+$. The firm maximizes its profit at prices $w \in \mathbb{R}_{++}^n$ for inputs and $q \in \mathbb{R}_{++}$ for output. The firm's profit maximization problem is

$$\max_{x \geq 0} qf(x) - wx$$

Production function f is strictly increasing but need not be differentiable. Let $x^*(q, w)$ denote the solution (input demand) and assume that x^* is a single valued function of q and w. State the definition of supermodular f and briefly explain the economic meaning of this definition. Show that if f is supermodular, then x^* is monotone nondecreasing in q.

Solution.

Definition 1. Function f is supermodular if for any $x, x' \in \mathbb{R}^n$ it holds

$$f(x \vee x') - f(x) \geq f(x') - f(x \wedge x')$$

where $x \wedge x' = (\min\{x_1, x_1'\}, \ldots, \min\{x_n, x_n'\})$ and $x \vee x' = (\max\{x_1, x_1'\}, \ldots, \max\{x_n, x_n'\})$.

Proof. For given w, define $F(x,q) = qf(x) - wx$. Based on Topkis theorem, since $\mathbb{R}^n_+$ is a lattice, if F is supermodular in x and has nondecreasing differences in $(x;q)$, then $x^*(q,w) = \operatorname{argmax}_{x\geq 0} F(x,q)$ is monotone nondecreasing in q; this is, $q < q'$ and $z \in x^*(w,q), z' \in x^*(w,q')$ imply $z \wedge z' \in x^*(w,q)$, and $z \vee z' \in x^*(w,q')$.

Function F has nondecreasing differences in $(x;q)$ if for any $x, x' \in \mathbb{R}^n$, $q, q' \in \mathbb{R}_{++}$ such that $x' \geq x$, $q' \geq q$, it holds $F(x',q') - F(x,q') \geq F(x',q) - F(x,q)$.

- F is supermodular in x:

$$\begin{aligned}
F(x \vee x', q) - F(x,q) &= qf(x \vee x') - w(x \vee x') - qf(x) + wx \\
&= q[f(x \vee x') - f(x)] - w[(\max\{x_1, x_1'\}, \ldots, \max\{x_n, x_n'\}) - x] \\
&\geq q[f(x') - f(x \wedge x')] - w[(\max\{x_1, x_1'\}, \ldots, \max\{x_n, x_n'\}) - x] \\
&= q[f(x') - f(x \wedge x')] - w[x' - (\min\{x_1, x_1'\}, \ldots, \min\{x_n, x_n'\})] \\
&= qf(x') - wx' - qf(x \wedge x') + w(x \wedge x') \\
&= F(x', q) - F(x \wedge x', q)
\end{aligned}$$

  where the third line follows from f being supermodular, and the fourth line from the fact that $x \vee x' + x \wedge x' = x + x'$ for any x, x'.

- F has nondecreasing differences in $(x;q)$:

$$\begin{aligned}
F(x', q') - F(x', q) &= q'f(x') - wx' - qf(x') + wx' \\
&= (q' - q)f(x') \\
&\geq (q' - q)f(x) \\
&\geq q'f(x) - wx - qf(x) + wx = F(x, q') - F(x, q)
\end{aligned}$$

  if $x' \geq x$, $q' \geq q$, thus $F(x',q') - F(x,q') \geq F(x',q) - F(x,q)$. □

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Supermodularity

QII.1 S2009

Consider a profit maximizing firm with single output and n inputs, with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ assumed strictly increasing, continuous (but possibly nondifferentiable), and $f(0) = 0$. Let $q \in \mathbb{R}_{++}$ be the price of output and $w \in \mathbb{R}^n_{++}$ be the vector of prices of inputs. The firm is taxed at rate $t > 0$ of its total cost. The firm's profit maximization problem is

$$\max_{x \geq 0} [qf(x) - wx - t(wx)].$$

Let $x^*(t)$ denote the profit maximizing vector of inputs (assumed unique) as function of tax rate t.

(a) State a definition of production function f being supermodular. State a criterion for supermodularity of f under an additional assumption that f is twice differentiable.

(b) Show that if f is supermodular, then input demand x^* is a nonincreasing function of t, that is, if $t' \geq t$, then $x^*(t') \leq x^*(t)$. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.

Solution

(a) **Definition 2.** Function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is supermodular if for all $x, x' \in \mathbb{R}^n_+$

$$f(x \vee x') - f(x) \geq f(x') - f(x \wedge x')$$

where $x \wedge x' = (\min\{x_1, x'_1\}, \ldots, \min\{x_n, x'_n\})$ and $x \vee x' = (\max\{x_1, x'_1\}, \ldots, \max\{x_n, x'_n\})$.

Claim 1. *A twice differentiable function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is supermodular if and only if for all $x \in \mathbb{R}^n_+$, all $i, j \in \{1, \ldots, n\}$*

$$\frac{\partial f}{\partial x_i \partial x_j}(x) \geq 0$$

(b) **Theorem 1** (Topkis)**.** *Let $S \subset \mathbb{R}^n$ be a lattice, let $T \subset \mathbb{R}^m$, let $F : S \times T \to \mathbb{R}$ and consider a problem $\max_{x \in S} F(x, t) \quad t \in T$. If F is supermodular in x, and has nonincreasing differences in (x, t), then $x^*(t) = \operatorname{argmax}_{x \in S} F(x, t)$ in monotone nonincreasing in t, that is $t < t'$ and $z \in x^*(t), z' \in x^*(t')$ imply that $z \wedge z' \in x^*(t')$ and $z \vee z' \in x^*(t)$.*

Function F has nonincreasing differences in (x, t) if for any $x, x' \in S$, $t, t' \in T$, such that $x \leq x'$, $t \leq tx'$ it holds $F(x', t') - F(x, t') \leq F(x', t) - F(x, t)$.
In our case let $F(x, t) = qf(x) - wx - t(wx)$.

- F is supermodular in x

$$\begin{aligned}
F(x \vee x', t) - F(x, t) &= qf(x \vee x') - (1 + t)w(x \vee x') - qf(x) + (1 + t)wx \\
&= q[f(x \vee x') - f(x)] - (1 + t)w[(x \vee x') - x] \\
&\geq q[f(x') - f(x \wedge x')] - (1 + t)w[(x \vee x') - x] \\
&= q[f(x') - f(x \wedge x')] - (1 + t)w[x' - (x \wedge x')] \\
&= qf(x') - (1 + t)wx' - qf(x \wedge x') + (1 + t)w(x \wedge x') \\
&= F(x', t) - F(x \wedge x', t)
\end{aligned}$$

  where the third line follows from f being supermodular, and the fourth from the fact that $x + x' = x \vee x' + x \wedge x'$ for any x, x'.

- F has noincreasing differences in (x, t)

$$\begin{aligned}
F(x', t') - F(x', t) &= qf(x') - (1 + t')wx' - qf(x') + (1 + t)wx' \\
&= -(1 + t')wx' + (1 + t)wx' \\
&= (t - t')wx' \\
&\leq (t - t')wx \\
&= -(1 + t')wx + (1 + t)wx \\
&= qf(x) - (1 + t')wx - qf(x) + (1 + t)wx \\
&= F(x, t') - F(x, t)
\end{aligned}$$

  if $t \leq t'$, $x \leq x'$, thus $F(x', t') - F(x, t') \leq F(x', t) - F(x, t)$.

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Risk aversion

QII.2 S2009

Consider an agent whose preferences over state-contingent consumption plans on a finite state space S have an expected utility representation with strictly increasing and twice-differentiable utility function $v : \mathbb{R} \to \mathbb{R}$ and probability measure π on 2^S.

Prove that the agent is risk averse if and only if $E[v(\tilde{z})] \geq E[v(\tilde{y})]$ for every $\tilde{y} : S \to \mathbb{R}_+$ and $\tilde{z} : S \to \mathbb{R}_+$ such that $E(\tilde{y}) = E(\tilde{z})$ and $\tilde{y}$ is more risky than $\tilde{z}$. Expected value E is taken with respect to probability measure π. Your definition of more risky should be stated in terms of cumulative distribution functions of $\tilde{y}$ and $\tilde{z}$. You may use the Theorem of Pratt without proving it.

Solution.

Definition 3. An agent with von Neumann Morgenstern utility function $v : \mathbb{R}_+ \to \mathbb{R}$ is risk averse if for every risky consumption plan $\tilde{c} : S \to R_+$ it holds $v(E(\tilde{c})) \geq E(v(\tilde{c}))$.

Definition 4. Let $\tilde{y}, \tilde{z}$ be two risky consumption plans $\tilde{y} : S \to R_+$, $\tilde{z} : S \to R_+$. Plan $\tilde{y}$ is more risky than $\tilde{z}$ if $E(\tilde{y}) = E(\tilde{z})$ and $\int_{-\infty}^{x} F_{\tilde{y}}(t)dt \geq \int_{-\infty}^{x} F_{\tilde{z}}(t)dt$ for every $x \in \mathbb{R}$.

Proof. Same as in Exercises II Q7 F2008. □

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Theorem of Pratt

QI.1 F2008

(a) State the Theorem of Pratt asserting equivalence of three ways of comparing risk aversion of agents whose preferences over risky claims have expected utility representation: Arrow-Pratt measure of risk aversion, risk compensation, and concave transformation of the von Neumann-Morgenstern utility function. Make sure that you clearly list all assumptions of the theorem.

(b) Prove the following two parts of the theorem you stated: (i) ranking according to Arrow-Pratt measure implies ranking according to concave transformation of utility function, (ii) ranking according to concave transformation of utility function implies ranking according to risk compensation.

(c) Give an example of two von Neumann-Morgenstern utility functions v_1 and v_2 such that neither v_1 is more risk averse than v_2 nor v_2 is more risk averse than v_1 in the sense of the Theorem of Pratt.

Solution

Definition 5. Let $v : \mathbb{R} \to \mathbb{R}$ be a twice differentiable von Neumann Morgenstern utility function. We define its Arrow-Pratt measure of risk aversion as $A(x) = -\frac{v''(x)}{v'(x)}$; and for any deterministic $w \in \mathbb{R}$ the risk compensation $\rho(w, \tilde{z})$ for an additional risky plan $\tilde{z}$ with $E(\tilde{z}) = 0$, by $E(v(w+\tilde{z})) = v(w-\rho(w, \tilde{z}))$.

(a) **Theorem 2.** *Let v_1, v_2 be two strictly increasing C^2 vNM utility functions. Then the following are equivalent*

   i. $A_1(x) \geq A_2(x)$, for all $x \in \mathbb{R}$

ii. $\rho_1(w,\tilde{z}) \geq \rho_2(w,\tilde{z})$*, for all* $w \in \mathbb{R}$*, all plans* $\tilde{z}$ *such that* $E(\tilde{z}) = 0$*.*

iii. there exists a concave function $f : \mathbb{R} \to \mathbb{R}$ *such that* $v_1(x) = f(v_2(x))$ *for all* $x \in \mathbb{R}$

(b) *Proof.*

(i) Since we are looking for $v_1(x) = f(v_2(x))$, let $t = v_2(x)$ and let $f(t) = v_1(v_2^{-1}(t))$. Then

$$v_1'(x) = f'(v_2(x))v_2'(x) \quad \Rightarrow \quad f'(t) = \frac{v_1'(v_2^{-1}(t))}{v_2'(v_2^{-1}(t))} > 0$$

and

$$v_1''(x) = f''(v_2(x))(v_2'(x))^2 + f'(v_2(x))v_2''(x)$$

from which

$$\begin{aligned} f''(t) &= \frac{v_1''(v_2^{-1}(t)) - f'(v_2(v_2^{-1}(t)))v_2''(v_2^{-1}(t))}{(v_2'(v_2^{-1}(t)))^2} \\ &= \frac{v_1''(v_2^{-1}(t)) - \frac{v_1'(v_2^{-1}(t))}{v_2'(v_2^{-1}(t))}v_2''(v_2^{-1}(t))}{(v_2'(v_2^{-1}(t)))^2} \\ &= \frac{v_1''(v_2^{-1}(t))}{v_1'(v_2^{-1}(t))}\frac{v_1'(v_2^{-1}(t))}{(v_2'(v_2^{-1}(t)))^2} - \frac{v_2''(v_2^{-1}(t))}{v_2'(v_2^{-1}(t))}\frac{v_1'(v_2^{-1}(t))}{(v_2'(v_2^{-1}(t)))^2} \\ &= \left(A_2(v_2^{-1}(t)) - A_1(v_2^{-1}(t))\right)\frac{v_1'(v_2^{-1}(t))}{(v_2'(v_2^{-1}(t)))^2} \leq 0 \end{aligned}$$

since by assumption $A_1(x) \geq A_2(x)$. Thus f is strictly increasing and concave.

(ii) By the definition of risk compensation

$$v_1(w - \rho_1(w,\tilde{z})) = E(v_1(w + \tilde{z}))$$

and since $v_1(x) = f(v_2(x))$ we also have

$$v_1(w-\rho_2(w,\tilde{z})) = f(v_2(w-\rho_2(w,\tilde{z}))) = f(E(v_2(w+\tilde{z}))) \geq E(f(v_2(w+\tilde{z}))) = E(v_1(w+\tilde{z}))$$

where the inequality comes from f being a concave function. Since $v_1$is strictly increasing these two imply $\rho_1(w,\tilde{z}) \geq \rho_2(w,\tilde{z})$. □

(c) Same as in Exercises II Q6 F2008.

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Weak Axiom of Profit Maximization

QI.2 F2008

Consider a finite set of observations $\{p^t, y^t\}_{t=1}^T$ of pairs of price vectors $p^t \in \mathbb{R}^l_{++}$ and production plans $y^t \in \mathbb{R}^l$. We say that production set $Y \subset \mathbb{R}^l$ profit-rationalizes this set of observations if $y^t \in Y$ and $p^t \cdot y^t = \max_{y \in Y} p^t \cdot y$ for every t.

(a) State the Weak Axiom of Profit Maximization (WAPM).

(b) Prove that a set of observations $(p^1, y^1), \ldots, (p^T, y^T)$ satisfies WAPM if and only if there exists a closed, convex production set Y that profit-rationalizes these observations.

Solution.

(a) Set of observations $\{p^t, y^t\}_{t=1}^T$ satisfies Weak Axiom of Profit Maximization (WAPM) if $p^i y^i \geq p^i y^j$ for every $i, j \in \{1, \dots, T\}$.

(b) *Proof.*

Assume that $(p^1, y^1), \dots, (p^T, y^T)$ satisfies WAPM.

Consider set Y which is a convex hull of $\{y^1 \dots, y^T\}$, that is let

$$Y = \left\{ y \in \mathbb{R}^l : \; \exists \alpha \in R_+^T \text{ such that } \sum_{t=1}^T \alpha_t = 1 \text{ and } y = \sum_{t=1}^T \alpha_t y^t \right\}$$

By construction then, this set is convex. Since Y is an image of a closed and bounded set $\Delta = \left\{ \alpha \in \mathbb{R}_+^T : \; \sum_{t=1}^T \alpha_t = 1 \right\}$ under a continuous function $f(\alpha) = \sum_{t=1}^T \alpha_t y^t$ we know that it is also closed and bounded.

Finally, consider arbitrary $y = \sum_{j=1}^T \alpha_j y^j \in Y$. Since by assumption for any $t, j \in \{1, \dots, T\}$ it holds that $p^t \cdot y^t \geq p^t \cdot y^j$, then multiplying this by α_j and summing across all $j's$ we obtain

$$\sum_{j=1}^T \alpha_j p^t \cdot y^t \geq \sum_{j=1}^T \alpha_j p^t \cdot y^j$$

$$p^t \cdot y^t \geq p^t \cdot \left(\sum_{j=1}^T \alpha_j y^j \right)$$

$$p^t \cdot y^t \geq p^t \cdot y$$

Thus $p^t \cdot y^t = \max_{y \in Y} p^t \cdot y$ for every t.

⇐ Assume that there exists a closed, convex production set Y that profit-rationalizes these observations.

By the definition of Y which profit-rationalizes $\{p^t, y^t\}_{t=1}^T$, it holds that $y^t \in Y$ and $p^t \cdot y^t = \max_{y \in Y} p^t \cdot y$ for every t. Then by taking $y^t = y^i$, $p^t = p^i$ and $y = y^j$ we have $p^i \cdot y^i \geq p^i \cdot y^j$. □

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Risk Aversion

QI.1 S2008

Consider two real-valued random variables y and z with the same expectations, $E(y) = E(z)$. Answer the following question (a):

(a) Show that if y is more risky than z, then $var(y) \geq var(z)$, where $var(y)$ is the variance of y.

Answer either one of questions (b) or (c), but not both.

(b) Show that, if z is more risky than y and y is more risky than z, then y and z have the same distribution, i.e., $F_y(t) = F_z(t)$ for every t, where F_y and F_z denote the cumulative distribution functions of y and z. You may assume that y and z take only finitely many values.

(c) Give an example of random variables y and z (with the same expectations) such that neither y is more risky than z nor z is more risky than y.

Solution.

(a) If y is more risky than z, then by definition $E(y) = E(z)$ and y second order stochastically dominates z. Also, we know that y second order stochastically dominates z if and only if for any nondecreasing concave continuous function g it holds $E(g(y)) \leq E(g(z))$. Let $a = E(z) = E(y)$ and $g(x) = -(x-a)^2$. Then $-var(y) = E(g(\tilde{y})) \leq E(g(\tilde{z})) = -var(z)$.

(b) Using the definition of the second order stochastic dominance in both directions we have that y and z satisfy for any $x \in \mathbb{R}$
$$\int_{-\infty}^{x} F_y(t)dt = \int_{-\infty}^{x} F_z(t)dt$$
Assume now that y, z take only finitely many values $y_1, \ldots, y_{n_y}$ and $z_1, \ldots, z_{n_z}$. Suppose that for some $1 \leq k \leq \min\{n_y, n_z\}$, it holds that $y_k \neq z_k$ and $y_j = z_j$, $j < k$, $Prob(y = y_j) = Prob(z = z_j)$, $j \leq k$. Without loss of generality suppose that $y_k \leq z_k$. Then for $x = (y_k + z_k)/2$
$$\int_{-\infty}^{x} (F_y(t) - F_z(t))dt = (z_k - y_k)/2Prob(y = y_k) \neq 0$$
which is a contradiction. Similarly suppose that for some $1 \leq k \leq \min\{n_y, n_z\}$, it holds that $y_j = z_j$, $j \leq k$ but $Prob(y = y_k) \neq Prob(z = z_k)$, $Prob(y = y_j) = Prob(z = z_j)$, $j < k$. Let $x = (y_k + \min\{z_{k+1}, y_{k+1}\})/2$ or $x = y_k + 1$ if $j = k$, then
$$\int_{-\infty}^{x} (F_y(t)dt - F_z(t))dt = (x - y_k)(Prob(y = y_k) - Prob(z = y_k)) \neq 0$$
which is again a contradiction. A similar contradiction arises if $n_y \neq n_z$.

(c) Same as in Exercises II Q5 F2008.

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Generalized Weak Axiom of Revealed Preference

QI.2 S2008

Consider a finite set $\{p^t, x^t\}_{t=1}^T$ of pairs of price vectors $p^t \in \mathbb{R}^l_{++}$ and consumption bundles $x^t \in \mathbb{R}^l_+$. Utility function $u : \mathbb{R}^l_+ \to \mathbb{R}$ is said to rationalize this set if $u(x^t) \geq u(x)$ for every $x \in \mathbb{R}^l_+$ such that $p^t x^t \geq p^t x$.

(a) State the Generalized Weak Axiom of Revealed Preference (GWARP) for $\{p^t, x^t\}_{t=1}^T$.

(b) Show that if a locally non-satiated utility function u rationalizes $\{p^t, x^t\}_{t=1}^T$, then the GWARP holds.

(c) Show that the assumption of local non-satiation in (b) cannot be dispensed with. That is, provide an example of a utility function that is locally satiated and rationalizes a set of pairs of prices and consumption bundles that violates the GWARP.

(d) State the Theorem of Afriat providing necessary and sufficient conditions for a set of pairs of prices and consumption bundles to be rationalized by a locally non-satiated utility function. A proof is not asked for.

Solution.

(a) The set of observations $\{p^t, x^t\}_{t=1}^T$ satisfies the Generalized Weak Axiom of Revealed Preference (GWARP) if for all s, t

$$p^t x^s \leq p^t x^t \quad \Rightarrow \quad p^s x^s \leq p^s x^t$$

(b) Since utility function u rationalizes $\{p^t, x^t\}_{t=1}^T$, $u(x^t) \geq u(x)$ for every $x \in \mathbb{R}_+^l$ such that $p^t x^t \geq p^t x$. Consider now $x \in \mathbb{R}_+^l$ such that $p^t x^t > p^t x$. By local nonsatiation $\exists \epsilon > 0$, $\exists y \in B_\epsilon(x)$ such that $p^t x^t \geq p^t y$ and $u(y) > u(x)$. Hence $u(x^t) \geq u(y) > u(x)$, and we have shown that if u rationalizes $\{p^t, x^t\}_{t=1}^T$, $u(x^t) > u(x)$ for every $x \in \mathbb{R}_+^l$ such that $p^t x^t > p^t x$.

Suppose now that $p^t x^s \leq p^t x^t$ but $p^s x^s > p^s x^t$. Then by previous results $u(x^t) \geq u(x^s)$ and $u(x^s) > u(x^t)$, which is a contradiction.

(c) Consider the example with $l = 2$, $u(x) \equiv 1$ and the set of observations with $T = 2$, $x^1 = (1, 2)$, $p^1 = (1, 2)$, $x^2 = (2, 1)$, $p^2 = (2, 1)$. Then $p^1 x^1 = p^2 x^2 = 5$, $p^1 x^2 = p^2 x^1 = 4$, hence $p^1 x^2 \leq p^1 x^1$ but $p^2 x^2 > p^2 x^1$, and so GWARP does not hold even though trivially $u(x^t) \geq u(x)$ for all x such that $p^t x \leq p^t x^t$.

(d) **Theorem 3** (Afriat)**.** *A set $\{p^t, x^t\}_{t=1}^T$ is rationalized by a locally non-satiated utility function if and only if these observations satisfy the Generalized axion of revealed preference (GARP); for all $t_1, \dots t_n, \in \{1, \dots, T\}$*

$$x^{t_1} R x^{t_2}, x^{t_2} R x^{t_3}, \dots, x^{t_{n-1}} R x^{t_n} \quad \Rightarrow \quad \neg(x^{t_n} P x^{t_1})$$

where $x^t R y$ if $p^t x^t \geq p^t y$, and $x^t P y$ if $p^t x^t > p^t y$.

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Risk aversion

QI.2 F2007

Consider two real-valued random variables Y and Z on a probability space. You may think about Y and Z as two contingent claims on a state space. You may assume that the state space is finite. Suppose that Y can take only one of two possible values y_1, y_2 with respective probabilities $\pi_1 > 0$ and $\pi_2 > 0$ such that $\pi_1 + \pi_2 = 1$. Suppose further that the expectations of Z conditional on $\{Y = y_1\}$ and $\{Y = y_2\}$ are zero, that is $E[Z|Y = y_1] = 0$ and $E[Z|Y = y_2] = 0$.

(a) Prove that $Y + Z$ is more risky (in the sense of second-order stochastic dominance) than Y.

(b) Prove that $Y + 2Z$ is more risky than $Y + Z$.

Solution

(a) Same as in Exercises II Q3 F2008.

(b) We need to show that $E(Y+2Z) = E(Y+Z)$, and that $E(g(Y+Z)) \geq E(g(Y+2Z))$ for every nondecreasing continuous concave function g.

$$\begin{aligned}
E(Y+2Z) &= \sum_{i=1}^{n}(y_1+2z_i)Prob(Z=z_i|Y=y_1)\pi_1 + \sum_{i=1}^{n}(y_2+2z_i)Prob(Z=z_i|Y=y_2)\pi_2 \\
&= \pi_1 E(y_1+2Z|Y=y_1) + \pi_2 E(y_2+2Z|Y=y_2) \\
&= \pi_1(y_1+2E(Z|Y=y_1)) + \pi_2(y_2+2E(Z|Y=y_2)) \\
&= \pi_1 y_1 + \pi_2 y_2 \\
&= E(Y) = E(Y+Z)
\end{aligned}$$

Note that $Y+Z = \frac{1}{2}(Y+2Z) + \frac{1}{2}(Y+0)$, then

$$\begin{aligned}
E(g(Y+Z)) &= E(g(\frac{1}{2}(Y+2Z) + \frac{1}{2}(Y+0))) \\
&\geq E(\frac{1}{2}g(Y+2Z) + \frac{1}{2}g(Y+0)) \\
&= \frac{1}{2}E(g(Y+2Z)) + \frac{1}{2}E(g(Y)) \\
&\geq \frac{1}{2}E(g(Y+2Z)) + \frac{1}{2}E(g(Y+Z))
\end{aligned}$$

thus $\frac{1}{2}E(g(Y+Z)) \geq \frac{1}{2}E(g(Y+2Z))$ or $E(g(Y+Z)) \geq E(g(Y+2Z))$. The second line uses the fact that g is concave and the last line results from part a.

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Risk aversion

QI.2 S2007

Consider an agent whose preferences over state-contingent consumption plans on a set of S states ($S > 1$) have an expected utility representation $E[v(c)]$, for some probabilities of states and a von Neumann-Morgenstern (or Bernoulli) utility function $v : \mathbb{R} \to \mathbb{R}$. Assume that utility function v is strictly increasing and twice-differentiable. Let $\rho(w, z)$ denote the risk compensation for risky claim $z \in \mathbb{R}^S$ with $E(z) = 0$ at risk-free initial wealth w. Let $A(w)$ denote the Arrow-Pratt measure of risk aversion at w.

(a) Prove that A is a weakly decreasing function of w if and only if risk compensation ρ is a weakly decreasing function of w for every z with $E(z) = 0$.

(b) Prove that the negative-exponential utility function $v(x) = -e^{-\alpha x}$ for $\alpha > 0$ is, up to an increasing linear transformation, the only von Neumann-Morgenstern utility function for which risk compensation $\rho(w, z)$ is independent of w for every z with $E(z) = 0$.

If you use the Theorem of Pratt, you need to state it clearly, but you are not asked to prove it.

Solution.

Definition 6. Let $v : \mathbb{R} \to \mathbb{R}$ be twice differentiable von Neumann Morgenstern utility function. The Arrow-Pratt measure of absolute risk aversion is $A(w) = -\frac{-v''(w)}{v'(w)}$.

Definition 7. Let $v : \mathbb{R} \to \mathbb{R}$ be a von Neumann Morgenstern utility function. For a fixed deterministic wealth $w \in \mathbb{R}$, the risk compensation for an additional risky plan z with $E(z) = 0$ is defined by $v(w - \rho(w, z)) = E(v(w + z))$.

Theorem 4 (Theorem of Pratt)**.** *Let v_1, v_2 be two twice differentiable strictly increasing von Neumann Morgenstern utility functions. Then the following are equivalent*

(i) $A_1(x) \leq A_2(x)$ *for all* $x \in \mathbb{R}$

(ii) $\rho_1(w, z) \leq \rho_2(w, z)$ *for all* $w \in \mathbb{R}$*, any risky consumption plan* z *with* $E(z) = 0$

(iii) there exists a concave, strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ *such that* $v_2(x) = f(v_1(x))$ *for any* $x \in \mathbb{R}$

(a) *Proof.* Fix $x \geq 0$ and consider $v_1(w) = v(w + x)$, and $v_2(w) = v(w)$.

$\Rightarrow$ Assume that A is a weakly decreasing function of w. Then

$$A_1(w) = -\frac{v_1''(w)}{v_1'(w)} = -\frac{v''(w+x)}{v'(w+x)} \leq -\frac{v''(w)}{v'(w)} = -\frac{v_2''(w)}{v_2'(w)} = A_2(w)$$

By Pratt's theorem this implies $\rho_1(w, z) \leq \rho_2(w, z)$.

$\Leftarrow$ Assume that the risk compensation ρ is a weakly decreasing function of w for every z with $E(z) = 0$. Then $w \leq w + x$ and using the assumption

$$\rho_1(w, z) = \rho(w + x, z) \leq \rho(w, z) = \rho_2(w, z)$$

By Pratt's theorem this implies $A_1(w, z) \leq A_2(w, z)$.

□

(b) *Proof.* Since $\rho(w, z)$ is independent of w we have that $\exists \alpha \in \mathbb{R}$, such that $A(x) = -\frac{v''(x)}{v'(x)} = \alpha$ for all $x \in \mathbb{R}$. This can be rewritten as a differential equation $\frac{d}{dx} \ln v'(x) = -\alpha$, the solution to which can be found from

$$\begin{aligned}
\ln v'(x) &= -\alpha x + C_1 \\
v'(x) &= e^{-\alpha x + C_1} \\
v(x) &= -\frac{1}{\alpha} e^{-\alpha x + C_1} + C_2 \\
&= -\frac{e^{C_1}}{\alpha} e^{-\alpha x} + C_2
\end{aligned}$$

Thus by choosing different C_1, C_2 we obtain different linear transformations of the utility function $u(x) = e^{-\alpha x}$, which all have coefficient of risk aversion independent of w. □

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Profit maximization

QI.1 F2006

Consider a competitive, profit-maximizing firm with production set $Y \subset \mathbb{R}^n$. Assume that set Y is closed and bounded, and that $0 \in Y$. Let π^* be the profit function and s^* the supply correspondence of the firm.

(a) Show that function π^* is continuous and convex.

(b) Show that correspondence s^* has closed graph.

(c) Assuming that π^* is differentiable at $p \in \mathbb{R}^n$, show that $D\pi^*(p) = s^*(p)$.

(d) Assuming that π^* is differentiable at $p \in \mathbb{R}^n$ and also that s^* is differentiable at p, prove the following comparative statics property of supply:

$$\frac{\partial s_i}{\partial p_i}(p) \geq 0$$

for every $i = 1, \ldots, n$.

You may use known results in mathematics without proofs, as long as you clearly state these results.

Solution

Definition 8. Let $Y \subset \mathbb{R}^n$ be a production set of a profit maximizing firm. The profit function is defined as

$$\pi^*(p) = \sup_{y \in Y} p \cdot y$$

and the supply correspondence as

$$s^*(p) = \{y^* \in Y : \ p \cdot y^* \geq p \cdot y, \ \forall y \in Y\}$$

Theorem 5 (Theorem of Maximum)**.** *Let $X \subseteq R^n$, $T \subseteq R^m$, $f : \mathbf{X} \times \mathbf{T} \to \mathbb{R}$, $\Gamma : \mathbf{T} \to \mathbf{2}^X$. Consider a problem*

$$h(t) = \max_{x \in \Gamma(t)} f(x, t)$$

and let

$$G(t) = \{x \in X : \ x \in \Gamma(t), \ f(x, t) = h(t)\}$$

If f is a continuous function, and Γ is a nonempty, continuous, compact-valued correspondence, then h is a continuous function and G is a nonempty, u.h.c., compact valued correspondence.

(a) *Proof.* Since Y is compact set and $p \cdot y$ is a continuous function, the supremum in the profit function is attained for some $y \in Y$ for every $p \in \mathbb{R}^n$. From the Theorem of Maximum it also follows that π^* is continuous function.

Let $p_1, p_2 \in \mathbb{R}^n$, and $\alpha \in [0, 1]$. Then

$$\begin{aligned}
\pi^*(\alpha p_1 + (1-\alpha)p_2) &= (\alpha p_1 + (1-\alpha)p_2) \cdot y^*_{\alpha p_1 + (1-\alpha)p_2} \\
&= \alpha p_1 y^*_{\alpha p_1 + (1-\alpha)p_2} + (1-\alpha)p_2 \cdot y^*_{\alpha p_1 + (1-\alpha)p_2} \\
&\leq \alpha p_1 y^*_{p_1} + (1-\alpha)p_2 y^*_{p_2} \\
&= \alpha \pi^*(p_1) + (1-\alpha)\pi^*(p_2)
\end{aligned}$$

where $y^*_p \in s^*(p)$, i.e. it is some $y \in Y$ that attains supremum in profit function for price p. $\square$

(b) For any sequence $(p_n, y_n) \subset \{(p,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n : \ y \in s^*(p)\}$ such that $(p_n, y_n) \to (\bar{p}, \bar{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n$ we have that for all $y \in Y$ it holds

$$p_n y_n \geq p_n y$$

Taking the limit on both sides

$$\lim_{n\to\infty}(p_n \cdot y_n) \geq \lim_{n\to\infty}(p_n \cdot y)$$

and since the dot product is continuous

$$\left(\lim_{n\to\infty} p_n\right) \cdot \left(\lim_{n\to\infty} y_n\right) \geq \left(\lim_{n\to\infty} p_n\right) \cdot y$$

or $\bar{p} \cdot \bar{y} \geq \bar{p} \cdot y$. Thus $s^*(p)$ has a closed graph.

(c) If π^* is differentiable, $s^*(p)$ is single valued. By definition

$$\pi^*(p) = p \cdot y^*(p) = \max_{y\in Y} p \cdot y = p \cdot s^*(p)$$

Then

$$D\pi^*(p) = D(ps^*(p)) = s^*(p) + pDs^*(p) = s^*(p)$$

since $pDs^*(p) = 0$ is the optimality condition for the supply correspondence.

(d) From part c. we know that $D\pi^*(p) = s^*(p)$. Thus if both π^* and s^* are differentiable

$$\frac{\partial s_i^*}{\partial p_j}(p) = \frac{\partial}{\partial p_j}\left(\frac{\partial \pi^*}{\partial p_i}(p)\right) = \frac{\partial^2 \pi^*}{\partial p_i \partial p_j}(p)$$

Since from part a. we know that π^* is convex $Ds^*(p) = D^2\pi^*(p)$ is a positive semidefinite matrix, and so all its diagonal elements are nonnegative. Therefore $\frac{\partial s_i}{\partial p_i}(p) \geq 0$.

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Risk aversion

QI.1 F2005

Let X and Z be two real-valued random variables on some probability space $(\Omega, \mathscr{F}, P)$. Suppose that $E(Z) = 0$.

(a) Consider the following statement: for every X and Z with $E(Z) = 0$, $X + Z$ is more risky than X. Show that this statement is false.

(b) Under what additional condition(s) on Z and/or X is the statement from part (a) true. Your conditions should not restrict X to be deterministic. Be as general as you can. Give an example of non-deterministic X and Z that satisfy your conditions.

(c) Prove your statement from part (b).

Solution

(a) Suppose that $E(X) = 0$ and $Z = -X$. Then $E(Z) = E(X) = 0$, moreover since $X + Z \equiv 0$ we have $E(X + Z) = 0$, and $E(g(X + Z)) = E(g(0)) = g(0) = g(E(X)) \geq E(g(X))$ for any nondecreasing, concave, continuous function g. Thus $X + Z$ second order stochastically dominates X, and so actually X is more risky than $X + Z$, not the opposite.

(b)

Claim 2. *If X, Z are two random variables with $E(Z) = 0$ and X, Z are independent, then $X + Z$ is more risky than X.*

(c) *Proof.* First, if $E(Z) = 0$, then $E(X + Z) = E(X) + E(Z) = E(X)$. Second, if X, Z are two random variables which are independent, then the joint probability distribution function is the product $dF_X(x)dF_Z(z)$ and for any nondecreasing, concave, continuous function g we have

$$
\begin{aligned}
E(g(X+Z)) &= \int\int g(x+z)dF_X(x)dF_Z(z) \\
&= \int \left(\int g(x+z)dF_X(x) \right) dF_Z(z) \\
&= \int E(g(x+z)|x)dF_X(x) \\
&\leq \int g(E(x+z)|x)dF_X(x) \\
&= \int g(x)dF_X(x) = E(g(x))
\end{aligned}
$$

where the third line is the result of Jensen's inequality. □

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Beth Allen

Pareto optimality

QI.1 S2009

Consider a pure exchange economy with two traders (indexed by subscripts 1 and 2) and l commodities. Suppose that both initial endowment vectors, e_1 and e_2, are strictly positive ($e_i \in \mathbb{R}^l_{++}, i = 1, 2$). Let $F = \{(x_1, x_2) \in \mathbb{R}^{2l}_+ | x_1 + x_2 = e_1 + e_2\}$ denote the set of feasible allocations (with no free disposal). Suppose that each trader $i = 1, 2$ has a preference relation $\succcurlyeq_i$ defined on F which is representable by a utility function $u_i : F \to \mathbb{R}$.

(a) Define what it means for u_i to represent $\succcurlyeq_i$.

(b) If there is a utility $u_i : F \to \mathbb{R}$ that represents $\succcurlyeq_i$, what assumption must $\succcurlyeq_i$ satisfy?

(c) What additional assumptions (be very precise) are needed for the first welfare theorem to hold in this economy?

(d) What minimal additional assumptions on $\succcurlyeq_1$ and $\succcurlyeq_2$ guarantee that u_1 and u_2 can always be chosen to represent $\succcurlyeq_1$ and $\succcurlyeq_2$ respectively such that at any Pareto optimal allocation $\hat{x} \in \mathbb{R}^{2l}_+, u_1(\hat{x}) < u_2(\hat{x})$?
Can you find an example of $\succcurlyeq_1$ and $\succcurlyeq_2$ on F such that there is a utility $\bar{u}_1 : F \to \mathbb{R}$ representing $\succcurlyeq_1$ for which there is no $\bar{\bar{u}}_2 : F \to \mathbb{R}$ representing $\succcurlyeq_2$ such that $\bar{u}_1(\hat{x}) < \bar{\bar{u}}_2(\hat{x})$ for every Pareto optimal allocation $\hat{x}$. Hint: Consider linear preferences of F such that $x' \sim_1 x''$ if and only if $x'_{11} + x'_{12} = x''_{11} + x''_{12}$ and $x' \sim_2 x''$ if and only if $x'_{21} + x'_{22} = x''_{21} + x''_{22}$ where $x \in F$ is written as $x = (x_{11}, x_{12}, x_{21}, x_{22})$ but suppose that person 1 is indifferent between consuming all of the economy's resources and consuming nothing.

Solution

(a) **Definition 9.** Function $u_i : F \to \mathbb{R}$ represents $\succcurlyeq_i$ if for any $x, y \in F$, $x \succcurlyeq_i y$ if and only if $u_i(x) \geq u_i(y)$.

(b) If there is a utility $u_i : F \to \mathbb{R}$ that represents $\succcurlyeq_i$, then $\succcurlyeq_i$ is a complete preorder.

(c) For the first welfare theorem to hold in this economy we only need $\succcurlyeq_i$ to be a complete preorder which is locally nonsatiated.

(d) UNFINISHED.

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Nonconvexities

QI.2 S2009

In general equilibrium theory, changing from a model with a finite number of traders to one with uncountably many (in fact, an atomless continuum of traders) can lead to better results on the existence of competitive equilibrium and its welfare properties.

Consider a pure exchange economy in which there are l commodities and each trader i has consumption set $X_i \subseteq \mathbb{R}^l_+$ and $e_i \in intX_i$, so that $e_i \gg 0$. For each complication below, explain whether changing from a finite number of traders to an atomless continuum leads to either better results or simpler/easier proofs for the existence of competitive equilibrium and the first and second fundamental theorems of welfare economics. Explain your reasoning briefly. You may assume that preferences $\succcurlyeq_i$ are strictly monotone complete continuous preorders defined on X_i.

(a) Preferences that are convex but not strictly convex.

(b) Nonconvex preferences.

(c) Nonconvex consumption sets that are closed (and bounded from below since $X_i \subseteq \mathbb{R}^l_+$).

(d) Consumption externalities.

Solution

(a) Existence theorem. If preferences are convex excess demand is a correspondence, not a single valued function. This however does not cause a problem, as there are theorems to prove that equilibrium exists even if excess demand is a correspondence.

First welfare theorem. First welfare theorem does not require any convexity assumption. With strictly monotone preferences the proof does not need to be changed significantly. We only need to introduce some technical measurability assumptions if there is a continuum of agents.

Second welfare theorem. If preferences that are convex but not strictly convex, the second any Pareto optimal allocation can still be supported as equilibrium with transfers.

(b) Existence theorem. If preferences are not convex excess demand is not convex valued. Thus existence theorem proof no longer works, since it is based on Kakutani's fixed point theorem, and the correspondence constructed in the proof will not be convex-valued if Z is not convex-valued. Changing the model to one with continuum of agents "convexifies" aggregate demand.

First welfare theorem. Same as in part a. First welfare theorem does not require any convexity assumption. With strictly monotone preferences the proof does not need to be changed significantly. We only need to introduce some technical measurability assumptions if there is a continuum of agents.

Second welfare theorem. With nonconvex preferences and finite number of consumers the second welfare theorem does not hold. We can construct an example where a Pareto optimal allocation in an economy with finite number of consumers can not be supported by any price vector. Changing the model to a continuum of consumers solves this problem.

(c) UNFINISHED.

(d) In the case of a negative consumption externality, the competitive equilibrium in the economy with finite number of agents may not be Pareto optimal.

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First welfare theorem

QII.1 F2007

This question applies to pure exchange economies with l commodities and n traders, $i = 1, \ldots, n$, each having initial endowment vector $e_i \in \mathbb{R}^l$ and preferences $\succcurlyeq_i$ which are assumed throughout to be continuous complete preorders on the consumption set $X_i \subseteq \mathbb{R}^l$.

(a) State the first welfare theorem.

(b) Prove the first welfare theorem.

(c) Does the conclusion of the first welfare theorem hold when the following complications separately (one at a time) are present?
Justify each answer by one of the following methods: pointing out that it doesn't affect your proof, explaining how your proof can be modified to encompass the complication, providing a counterexample to the conclusion of the first welfare theorem when this one complication is present, or explaining precisely how the complication prevents your proof from being modified to demonstrate that the first welfare theorem holds despite the complication. The complications are as follows:

   (i) preferences that are convex but not strictly/strongly convex

   (ii) preferences that are weakly convex but not convex

   (iii) preferences that are nonsatiated but not locally nonsatiated

   (iv) preferences that are strictly monotone but consumption sets X_i are not necessarily convex

(d) Briefly discuss (i.e., in an essay of 50-300 words) the economic significance of the first welfare theorem.

Solution

(a) **Theorem 6.** *Consider a pure exchange economy* $\mathcal{E} = (I, (X_i, \succcurlyeq_i, e_i)_{i \in I})$ *where for all* $i \in I = \{1, \dots, n\}$ $\succcurlyeq_i$ *is a complete, locally nonsatiated preorder on* $X_i = \mathbb{R}^l_+$. *If* $x^* \in R^{ln}$ *is allocation in the competitive equilibrium* (x^*, p^*), *then* x^* *is strongly Pareto optimal.*

(b) *Proof.* Suppose not, then there exists a feasible allocation $\hat{x} \in R^{ln}$ such that for all $i \in I$, $\hat{x}_i \succcurlyeq_i x^*_i$ and $\exists i' \in I$ such that $\hat{x}_{i'} \succ_{i'} x^*_{i'}$. Since $\succcurlyeq_i$ are locally nonsatiated we know that for all $i \in I$ it holds $p^* x^*_i = p^* e_i$ and also $p^* \hat{x}_i \geq p^* e_i$, $p^* \hat{x}_{i'} > p^* e_{i'}$. Taking the sum for across i we obtain $\sum_{i \in I} p^* \hat{x}_i > \sum_{i \in I} p^* e_i$ or $p^* \sum_{i \in I} \hat{x}_i > p^* \sum_{i \in I} e_i$. But since $\hat{x}$ is feasible $\sum_{i \in I} \hat{x}_i \leq \sum_{i \in I} e_i$ and so $p^* \sum_{i \in I} \hat{x}_i \leq p^* \sum_{i \in I} e_i$, which is a contradiction. □

(c) If preferences $\succcurlyeq_i$ are

   (i) convex but not strictly convex - this does not affect FWT in any way, convexity assumption is not needed, as long as preference are nonsatiated, if they are convex they are also they locally nonsatiated.

   (ii) weakly convex but not convex - this may cause a problem since convex preferences allow for 'thick indifference curves', and in that case $\exists \epsilon > 0$ such that $x \sim y$ for $||x - y|| < \epsilon$. Thus preferences are not locally nonsatiated. As result $p^* x^*_i = p^* e_i$ may not hold, and also not $p^* x_i \geq p^* e_i$ for $x_i \succcurlyeq_i x^*_i$. But this is necessary for the proof to go through.

   (iii) nonsatiated but not locally nonsatiated - as above, $\exists \epsilon > 0$ such that $x \sim y$ for $||x - y|| < \epsilon$. Thus preferences are not locally nonsatiated. As result $p^* x^*_i = p^* e_i$ may not hold, and also not $p^* x_i \geq p^* e_i$ for $x_i \succcurlyeq_i x^*_i$. But this is necessary for the proof to go through.

   (iv) strictly monotone but consumption sets X_i are not necessarily convex - first welfare theorem might not hold. For example consider a two goods, two consumer economy with $X_i = \mathbf{N}^2_+$, endowments $e_1 = (1, 0), e_2 = (0, 1)$, and Cobb-Douglas preferences $u_i(x) = x_1^{0.5} x_2^{0.5}$ for

$i = 1, 2$. Then for any $p^* \gg 0$, (e_1, e_2) is equilibrium allocation, but is not Pareto optimal. Utility of one agent can be increased by giving him $(1, 1)$ without decreasing the utility of the other agent since $u_i((1, 1)) = 1$ and $u_i((0, 1)) = u_i((1, 0)) = u_i((0, 0)) = 0$.

Another example would be an economy with one good and two consumers with $X_i = \mathbf{N}_+$, endowments $e_1 = 0.5, e_2 = 0.5$, and preferences on X_i represented by a utility function $u_i(x) = x$. The equilibrium here is again not Pareto optimal since $u_1 = u_2 = 0$, and the utility of one agent can be increased by giving him 1 without decreasing the utility of the other agent since $u_i(1) = 1$ and $u_i(0) = u_i(0.5) = 0$. (Notice however that in this example we assume that the commodity set is $\mathbb{R}_+$, and only consumption set are not convex.)

(d) UNFINISHED

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Sonnenschein conjecture

QII.2 F2007

Consider a pure exchange economy with l commodities and n consumers, $i = 1, \ldots, n$, each having initial endowment vector $e_i \in \mathbb{R}^l$ and preferences $\succcurlyeq_i$ defined on the consumption set R^l_+. Each $\succcurlyeq_i$ is assumed to be a continuous complete preorder which satisfies strict convexity and strict monotonicity.

(a) What can be said about the aggregate excess demand Z of this economy? (i.e., state the theorem of Sonnenschein et. al. and be sure to specify the domain and range of the mapping Z.)

(b) For each property of Z you state in part (a), identify which assumption(s) on preferences are needed for the property and then prove that the assumption(s) you identify imply the property.

(c) Briefly discuss (i.e., in an essay of 50-300 words) the economic significance of the characterization of aggregate excess demand.

Solution

(a) **Theorem 7** (Sonnenschein-Mantel-Debreu Theorem)**.** *Let* $\Delta_\epsilon = \{p \in \Delta : p_k \geq \epsilon \; \forall k \in \{1, \ldots, l\}\}$*. Let* $Z : \Delta \to \mathbb{R}^l$ *be a function which is homogeneous of degree 0, continuous and satisfies Walras law* $pZ(p) = 0$*. Then for any* $\epsilon > 0$ *there exists a pure exchange economy* $\mathcal{E} = (l, (\mathbb{R}^l_+, \succcurlyeq_i, e_i)_{i \in \{1,\ldots,l\}})$*, where* $e_i \in \mathbb{R}^l_+$ *and* $\succcurlyeq_i$ *is a complete, monotone, strictly convex and continuous preorder on* $\mathbb{R}^l_+$*, such that the aggregate excess demand of this economy on* Δ_ϵ *is* Z*.*

(b) **Claim 3.** *If for all* $i \in \{1, \ldots, n\}$ *preferences* $\succcurlyeq_i$ *are a complete preorder with closed upper contour sets* $U_i(x) = \{y \in X_i : y \succcurlyeq_i x\}$*, then* Z *is homogenous of degree 0 on* Δ_ϵ*.*

Proof. Budget set $B(p, e_i)$ is a compact set if $p \gg 0$ and $e_i \geq 0$. Then, if $\succcurlyeq_i$ are a complete preorder with closed upper contour sets $U_i(x) = \{y \in X_i : y \succcurlyeq_i x\}$, the demand correspondence $x^*_i(p, e_i)$ is well defined. Then, if $x \in x^*_i(p, e_i)$ it holds that $px \leq pe_i$ and $x \succcurlyeq_i y$, for all $y \in B(p, e_i)$, and so for any $\lambda > 0$, $\lambda px \leq \lambda pe_i$, and $x \succcurlyeq_i y$, for all $y \in B(\lambda p, e_i)$ because $B(p, e_i) = \{x \in X_i : px \leq pe_i\} = \{x \in X_i : \lambda px \leq \lambda pe_i\} = B(\lambda p, e_i)$. This means that $x^*_i(p, e_i) - e_i$ is HOD0 in p on Δ_ϵ for all $i \in \{1, \ldots, n\}$ and thus also Z is HOD0 on Δ_ϵ. □

Claim 4. *If for all* $i \in \{1, \ldots, n\}$ *preferences* $\succcurlyeq_i$ *are a complete, continuous, strictly convex preorder then* Z *is continuous on* Δ_ϵ*.*

Proof. If $\succcurlyeq_i$ are a complete, continuous preorder they can be represented by a continuous utility function. It can be also shown that $B(\cdot, e_i)$ is correspondence which is continuous, (it is both uhc and lhc), and for $p \in \Delta_\epsilon$ it is also compact valued and nonempty valued. By the Maximum Theorem, $x_i^*(\cdot, e_i)$ is then an nonempty, uhc, compact valued correspondence. If in addition $\succcurlyeq_i$ are strictly convex then $x_i^*(\cdot, e_i)$ is also single valued, which combined with uhc implies that it is continuous. This means that $x_i^*(\cdot, e_i) - e_i$ is continuous on Δ_ϵ for all $i \in \{1, \ldots, n\}$ and thus also Z is continuous on Δ_ϵ. □

Claim 5. *If for all $i \in \{1, \ldots, n\}$ preferences $\succcurlyeq_i$ are a complete, lns preorder with closed upper contour sets $U_i(x) = \{y \in X_i : y \succcurlyeq_i x\}$, then Z satisfies Walras law that is $pZ(p) = 0$, on Δ_ϵ.*

Proof. If $\succcurlyeq_i$ are a complete, preorder with closed upper contour sets $U_i(x) = \{y \in X_i : y \succcurlyeq_i x\}$, the demand correspondence $x_i^*(p, e_i)$ is well defined on Δ_ϵ. With lns for any $x \in x_i^*(p, e_i)$ it also holds that $px = pe_i$. Then $pZ(p) = p\sum_{i=1}^n (x_i^*(p, e_i) - e_i) = \sum_{i=1}^n p(x_i^*(p, e_i) - e_i) = 0$. □

Recall also that if $\succcurlyeq_i$ are strictly monotone $\Rightarrow$ $\succcurlyeq_i$ are monotone $\Rightarrow$ $\succcurlyeq_i$ are lns.

(c) UNFINISHED

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Generic approach

QII.1 S2007

This question concerns smooth pure exchange economies with n traders, $i = 1, 2, \ldots, n$ and l commodities. Each consumer i has preferences $\succcurlyeq_i$ (assumed to be complete preorders) and initial endowment $e_i \in \mathbb{R}^l_+$.

(a) What additional assumptions on preferences and endowments guarantee that aggregate excess demand $Z : \Delta \to \mathbb{R}^l$ is continuously differentiable (C^1), where $\Delta = \{p \in \mathbb{R}^l_{++} : \sum_{k=1}^l p_k = 1\}$?

(b) Define the generic approach and discuss its economic interest.

(c) If Z is C^1, what properties does the equilibrium price correspondence satisfy? Briefly discuss the economic implications of these properties.

(d) If the economies satisfy your assumptions in (a) so that Z is C^1, what properties are satisfied for generic profiles $e = (e_1, \ldots, e_n) \in \mathbb{R}^{ln}_+$ of initial endowments? Briefly discuss the economic implications of this.

(e) How would your answer to (a) be changed if Z were merely required to be continuous (C^0)?

(f) Then how would your answer to (c) change if Z were assumed only to be C^0? Prove your claim.

(g) Given your answer to (e) [in place of (a)], how would your answer to (d) be changed if Z were assumed to be C^0 but not necessarily C^1?

Solution

(a) **Claim 6.** *Suppose that for all $i \in \{1, \ldots, n\}$ preferences $\succcurlyeq_i$ are such that they can be represented by a utility function $u_i : \mathbb{R}^l_+$ wich is* *(a) twice continuously differentiable on* $\mathbb{R}^l_{++}$*,*

(b) strictly differentiably monotone i.e. $\forall x \in R^l_+ \; Du_i(x) \gg 0$

(c) strictly differentiably concave i.e. $\forall x \in R^l_+$*,* $v^T D^2 u_i(x) v < 0$ *for all* $v \neq 0$*,* $D(x)v = 0$

(d) satisfies boundary condition i.e. if $u_i(x) \geq u_i(\hat{x})$ *for some* $\hat{x} \gg 0$*, then* $x \gg 0$

and

(e) $e_i \gg 0$

then the aggregate excess demand function Z *in this economy is continuously differentiable.*

(b) UNFINISHED

(c) UNFINISHED

(d) UNFINISHED

(e) UNFINISHED

(f) UNFINISHED

(g) UNFINISHED

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First welfare theorem

QII.2 S2007

This question concerns the first fundamental theorem of welfare economics. First consider a classical pure exchange economy with l commodities and n traders, $i = 1, 2, \ldots, n$, with initial endowment $e_i \in \mathbb{R}^l_+$ and preferences $\succcurlyeq_i$ assumed to be a continuous complete preorder on the consumption set $\mathbb{R}^l_+$

(a) State the first welfare theorem.

(b) Prove the first welfare theorem.

(c) Briefly interpret and discuss the economic significance of your assumptions in (a).

(d) Now suppose that, instead of n traders, the economy has a continuum of traders. Define feasible allocation and Pareto optimal allocation. State a version of the first welfare theorem that holds in such economies and explain your reasoning, including the changes (if any) that are required to the theorem you stated in (a).

(e) Finally, suppose that instead of uncountably many agents as in part (d), we consider the first welfare theorem with countably many agents. Specifically, consider an overlapping generations pure exchange economy. It is known that the first welfare theorem fails to be true in classical overlapping generations economies. What assumptions (relative to your answers to (a) and (d)) are violated in this case (beyond the obvious fact that now we have countably many traders)? Explain your answer.

Solution

(a)

Theorem 8. *Consider a pure exchange economy $\mathcal{E} = (I, (X_i, \succcurlyeq_i, e_i)_{i \in I})$ where for all $i \in I = \{1, \ldots, n\}$ $\succcurlyeq_i$ is a complete, locally nonsatiated preorder on $X_i = \mathbb{R}^l_+$. If $x^* \in R^{ln}$ is allocation in the competitive equilibrium (x^*, p^*), then x^* is strongly Pareto optimal.*

(b) *Proof.* Suppose not, then there exists a feasible allocation $\hat{x} \in R^{ln}$ such that for all $i \in I$, $\hat{x}_i \succcurlyeq_i x^*_i$ and $\exists i' \in I$ such that $\hat{x}_{i'} \succ_{i'} x^*_{i'}$. Since $\succcurlyeq_i$ are locally nonsatiated we know that for all $i \in I$ it holds $p^* x^*_i = p^* e_i$ and also $p^* \hat{x}_i \geq p^* e_i$, $p^* \hat{x}_{i'} > p^* e_{i'}$. Taking the sum for across i we obtain $\sum_{i \in I} p^* \hat{x}_i > \sum_{i \in I} p^* e_i$ or $p^* \sum_{i \in I} \hat{x}_i > p^* \sum_{i \in I} e_i$. But since $\hat{x}$ is feasible $\sum_{i \in I} \hat{x}_i \leq \sum_{i \in I} e_i$ and so $p^* \sum_{i \in I} \hat{x}_i \leq p^* \sum_{i \in I} e_i$, which is a contradiction. □

(c) Completeness assumption means that for any two bundles x, y agents is able to tell, whether shklee likes one or the other more, or likes both of them equally.

Local nonsatiation assumption means that for any bundle x we can find a budle y, that is arbitrarily close to x and is preferred to x. Thus satiation is never reached.

(d) Definitions based on Aumann (1966):

The commodity space is $X = \mathbb{R}^l_+$, the set of consumers is closed unit interval $I = [0, 1]$. An allocation is a Lebesgue integrable function $x : I \to X$. Initial endowments is a function $e : I \to X$, and we assume that $\int_I e(i) di >\gg 0$. An allocation x is feasible if $\int_I x(i) di \leq \int_I e(i) di$.

For any $i \in I$ preferences $\succcurlyeq_i$ are a binary relation on $\mathbb{R}^l_+$, which is strictly monotone, continuous and measurable, i.e. for any allocations x, y, the set $\{i \in I : x(i) \succcurlyeq_i y(i)\}$ is Lebesgue measurable in I.

Definition 10. An allocation x is Pareto optimal, if there does not exists any feasible allocation x', such that $x'(i) \succcurlyeq_i x(i)$ for all $i \in I$, and $x'(i) \succ_i x(i)$ for all $i \in S$, where $S \subseteq I$ has a nonzero Lebesgue measure.

Definitions based on Hildenbrand (1969), adopted for a pure exchange economy:

Definition 11. A commodity space is R^l. Let $(A, \mathscr{A}, \mu)$ be a measure space, where element of A are called economic agents, $\mathscr{A}$ are coalitions of agents, and $\mu(M)$ is the measure of agents in M, for any $M \in \mathscr{A}$. Let $X : A \to \mathbb{R}^l$ be the consumption set correspondence. To all $a \in A$ assign a binary relation $\succcurlyeq_a$ over $X(a)$. Let $\omega \in \mathbb{R}^l$ be the total resources of economy. Then $\mathcal{E} = \{(A, \mathscr{A}, \mu), X, \succcurlyeq, \omega\}$ defines a pure exchange economy.

Definition 12. Denote $\mathscr{L}_X$ the set of μ-integrable functions $f : A \to \mathbb{R}^l$ such that $f(a) \in X(a)$ for μ-almost every in $a \in A$. Then $\int_M f d\mu$ is the commodity vector allocated to coalition M under f, and $\{\int_M f d\mu : f \in \mathscr{L}_X\}$ is the set of all commodity vectors that can be allocated to coalition M. For $\omega \in \mathbb{R}^l$ denote further $\mathscr{L}_{X,\omega} = \{f \in \mathscr{L}_X : \int_A f d\mu = \omega\}$ the set of feasible allocations of ω over A.

Definition 13. Consider a pure exchange economy $\mathcal{E} = \{(A, \mathscr{A}, \mu), X, \succcurlyeq, \omega\}$. An allocation $f \in \mathscr{L}_{X,\omega}$ is Pareto optimal, if there does not exists any allocation $g \in \mathscr{L}_{X,\omega}$, such that there exists a coalition $M \subset \mathscr{A}$ with $\mu(M) > 0$ for which $g(a) \succ_a f(a)$ for all $a \in M$, and $g(a) \succcurlyeq_a f(a)$ for almost every $a \in A \setminus M$.

Definition 14. Consider a pure exchange economy $\mathcal{E} = \{(A, \mathscr{A}, \mu), X, \succcurlyeq, \omega\}$. Given a price vector $p \in \mathbb{R}^l$, an allocation $f \in \mathscr{L}_{X,\omega}$ is said to be an equilibrium relative to p, if almost everywhere in A, $z \in X(a)$ and $z \succ_a f(a)$ imply $p \cdot z > p \cdot f(a)$, that is $f(a)$ is maximal of $\succcurlyeq_a$ in the set $\{x \in X(a) : p \cdot x \leq p \cdot f(a)\}$.

Theorem 9 (First welfare theorem)**.** *Let $\mathcal{E}$ be a pure exchange economy where for almost every $a \in A$, $z \in X(a)$ and $\{x \in X(a) : x \succ_a z\} \neq \emptyset$ implies $\{x \in X(a) : x \succcurlyeq_a z\} \subseteq cl\{x \in X(a) : x \succ_a z\}$. Then an equilibrium f relative to a price vector p where almost all consumers are not satiated is Pareto optimal.*

Theorem 10 (Second welfare theorem)**.** *Let $\mathcal{E}$ be a pure exchange economy where*

i. for each $B \in \mathscr{A}$ with $\mu(B) > 0$, $\exists M \in \mathscr{A}$ such that $M \subset B$ and $\mu(B) > \mu(M) > 0$,

ii. for almost every $a \in A$, $z \in X(a)$ and $\{x \in X(a) : x \succ_a z\} \neq \emptyset$ implies $\{x \in X(a) : x \succcurlyeq_a z\} \subseteq cl\{x \in X(a) : x \succ_a z\}$,

iii. for all $f \in {}_{X,\omega}$

a. $\{(a, x) \in A \times X(a) : x \succ_a f(a)\} \in \mathscr{A}_\mu \times \mathscr{B}_l$

b. $\{(a, x) \in A \times X(a) : x \succcurlyeq_a f(a)\} \in \mathscr{A}_\mu \times \mathscr{B}_l$

where $\mathscr{B}_l$ is the Borel σ-algebra in $\mathbb{R}^l$, and $\mathscr{A}_\mu$ is the completion of $\mathscr{A}$ relative to μ.

iv. almost everywhere in A, $X(a)$ is convex

v. almost everywhere in A, for every $z \in Z(a)$ the set $\{x \in X(a) : x \succ_a z\}$ is open in $X(a)$

Then, if $f\mathscr{L}_{X,\omega}$ is such that $f(a)$ is non satiated for almost every $a \in A$, there exists a price vector $p \in \mathbb{R}^l$, $p \neq 0$ such that (f, p) is an equilibrium.

(e) The competitive equilibrium in an overlapping generations model may not be Pareto optimal, there is double infinity problem - there is infinite number of traders and also infinite number of commodities, and the proof of first welfare theorem does not go through since $\sum_{i\in I}\sum_{i\in L} p_l e_{il}$ is not finite.

When going from a countably many agents to a continuum of agents the definition of Pareto optimality is changed. In the case of a continuum of agents an allocation can be Pareto optimal even if a countably infinite number of agents (e.g. represented by rational numbers on the interval A) can be made better off - this set has Lebesgue measure zero.

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Nonconvexities

QII.1 F2006

Consider a pure exchange economy with commodity space $\mathbb{R}^l$ (so that there are l perfectly divisible commodities) and n traders $i = 1, \ldots, n$, each having consumption set $X_i \subseteq \mathbb{R}^l_+$ (assume that each X_i is closed), initial endowment vector $e_i \in X_i$, and preferences $\succcurlyeq_i$ (defined on X_i) which are assumed to be a complete continuous preorder.

(a) Define competitive equilibrium for this economy.

Assume that, for all $i = 1, \ldots, n$, $\succcurlyeq_i$ is locally nonsatiated.

(b) Define locally nonsatiated. The remainder (and major portion) of this question concerns the statement "Nonconvexities cause problems for the existence of competitive equilibrium because they result in discontinuities." It is essentially a "true or false and explain why" question, but you are asked to give a precise explanation.

(c) Identify the various ways in which the nonconvexities can arise in this economy. Briefly discuss the economic interpretation of each.

(d) For EACH possible type of nonconvexity, is the statement true or false?

- If it is true, clearly explain why: show (perhaps by a clear example or a clearly drawn picture) how the discontinuity arises AND then explain how the discontinuity can lead to nonexistence of equilibrium.
- If the statement if false, explain why: state and prove your continuity claim (or give a counterexample, clear picture, or precise explanation) and then either explain where this continuity is used in a standard proof for the existence of equilibrium or explain why existence of equilibrium doesn't require such continuity.

(e) For EACH possible type of nonconvexity, do the resulting problems (if any) with nonexistence of equilibrium disappear in a large economy? Explain.

Solution.

(a) **Definition 15.** A competitive equilibrium is $(p^*, x^*) \in \mathbb{R}^l_+ \times R^{ln}_+$ such that

(i) $x_i^* \in x_i^(p^*, e_i)$ for all $i \in \{1, \ldots, n\}$ where $x_i^*(p^*, e_i) = \{x_i \in X_i : p^* x_i \leq p^* e_i, x_i \succcurlyeq_i x_i', \forall x_i' \in X_i$ such that $p^* x_i' \leq p^* e_i\}$

(ii) $\sum_{i=1}^n x_i^* \leq \sum_{i=1}^n e_i^*$

preorder $\succcurlyeq_i$ on $X_i \subseteq \mathbb{R}^l_l$ is locally nonsatiated if $\forall x \in X_i$, $\forall \epsilon > 0$ $\exists y \in X_i$ such that $||x - y|| < \epsilon$ and $y \succ_i x$.

(b) **Definition 16.** A preorder $\succcurlyeq_i$ on $X_i \subseteq \mathbb{R}^l_l$ is locally nonsatiated if $\forall x \in X_i$, $\forall \epsilon > 0$ $\exists y \in X_i$ such that $||x - y|| < \epsilon$ and $y \succ_i x$.

(c) Nonconvexities in the pure exchange economy can be a result of either noncovex preferences or nonconvex consumption sets. Nonconvex preferences can arise for example as a result of consumption goods that can not be consumed together, thus the agent prefers extremes rather than combinations of goods. Noncovex consumption sets reflect the nondivisible goods.

(d) i. If preferences are not convex excess demand is not convex valued. Thus existence theorem proof no longer works, since it is based on Kakutani's fixed point theorem, and the correspondence constructed in the proof will not be convex-valued if Z is not convex-valued.

Consider a two agent economy where one agent has preferences represented by a utility function $u_1(x_1, x_2) = x_1^2 + x_2^2$ and the other one $u_1(x_1, x_2) = \min_{x_1, x_2}$, with initial endowments $e_1 = e_2 = (1, 1)$. Then the demand of the first agent is

$$x_1^*(p, e_1) = \begin{cases} \left(\frac{1}{p_1}, 0\right) & \text{if } p_1 < p_2 \\ \{(2,0), (0,2)\} & \text{if } p_1 = p_2 \\ \left(0, \frac{1}{p_2}\right) & \text{if } p_1 > p_2 \end{cases}$$

and the demand of the second agent is

$$x_2^*(p, e_1) = \{(1, 1)\}$$

For any $p = (p_1, p_2)$ then $0 \notin Z(p) = \sum_{i=1}^2 (x_i^*(p, e_1) - e_i)$.

ii. UNFINISHED

(e) If preferences are not convex, changing the model to one with continuum of agents ”convexifies” aggregate demand, thus solving the problem with nonexistence of equilibria.

The problem with nonconvex consumption sets is not affected and is still present even in a model with a continuum of agents.

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Ichiro Obara

Second welfare theorem

QII.1 F2008

For a pure exchange economy with l commodities and n traders,

(a) State the second welfare theorem (also known as the second fundamental theorem of welfare economics),

(b) Prove the theorem

(c) Briefly compare its statement and proof to the analogous part of the Debreu-Scarf Theorem.

Solution

(a) **Theorem 11.** *Consider a pure exchange economy $\mathcal{E} = (I, (X_i, \succcurlyeq_i, e_i)_{i \in I})$ where for every $i \in I = \{1, \ldots, n\}$ $\succcurlyeq_i$ is a complete, strictly monotone, strictly convex and continuous preorder on $X_i = \mathbb{R}^l_+$. If $x^* \in R^{ln}$ is a Pareto optimal allocation, then if $e^* = x^*$ is a redistribution of endowments, there exists $p^* \in \mathbb{R}^l_+$, $p^* \neq 0$ such that (e^*, p^*) is a unique competitive equilibrium given these endowments.*

(b) *Proof.*
Step 1. Construct a price vector p^* given x^*.

i. let $V_i = \{x \in \mathbb{R}^l_+ : x \succ_i x_i^*\}$ which are open and strictly convex sets (since $\succ_i$ are continuous and strictly convex)
ii. let $V = \sum_{i=1}^n V_i = \{x \in \mathbb{R}^l_+ : x = \sum_{i=1}^n x_i,\ x_i \in V_i,\ i = 1, \ldots, n\}$ which is also open and strictly convex set
iii. let $e = \sum_{i=1}^n x_i^*$ which is a trivially convex set
iv. $\{e\} \cap V = \emptyset$, if this was not the case and $\hat{x} = e$ for some $\hat{x} \in V$, then this would imply that there exists $(\hat{x}_1, \ldots, \hat{x}_n)$ such that $\sum_{i=1}^n \hat{x}_i = \hat{x} = e = \sum_{i=1}^n x_i^*$ and $\hat{x}_i \succ_i x_i^*$ for all i, which is a contradiction since x^* is Pareto optimal
v. be separating hyperplane theorem, $\exists p^* \in \mathbb{R}^l_+$, $p \neq 0$ and $r \in \mathbb{R}$ such that $p^* x \geq r \geq p^* e$ for all $x \in V$

Step 2. Show that $p^* \geq 0$

Consider $\hat{x}_i = x_i^* + u_k/n$ where $u_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the k-th unit basis vector in $\mathbb{R}^l_+$. By strict monotonicity of $\succcurlyeq_i$ we have $\hat{x}_1 \succ_i x_1^*$ and for $\hat{x} = \sum_{i=2}^I \hat{x}_i$, $\hat{x} \in V$ so that $p^* \hat{x} \geq p^* \hat{e}$ from which $p_k^* \geq 0$ for all $k \in \{1, \ldots, l\}$.

Step 3. Since $x_i* \in B(p^*, x_i*)$, to finish the proof we only need show that $\hat{x}_i \succ_i x_i^* \Rightarrow p^* \hat{x}_i > p^* x_i^*$.

i. If $\hat{x}_i \succ_i x_i^*$ then $p^* \hat{x}_i \geq p^* x_i^*$:
Since $\succcurlyeq_i$ are continuous and strictly monotone, $\exists \epsilon > 0$ and $k \in \{1, \ldots, l\}$, such that for

$\bar{x}_i = \hat{x}_i - \epsilon u_k$ we have $\hat{x}_i \succ_i \bar{x}_i \succ_i x_i^*$. By strict monotonicity of $\succsim_j$ for $\bar{x}_j = \hat{x}_j + \frac{\epsilon}{n-1} u_k$ we also have $\bar{x}_j = \succ_j x_j^*$ for $j \in I, j \neq i$. Then $\bar{x}_i \in V_i$, $\bar{x} = \sum_{i=1}^n \bar{x}_i \in V$ and so

$$p^* \sum_{j=1}^n x_j^* = p^* e \leq p^* \bar{x} = p^* \sum_{j=1}^n \bar{x}_j = p^* \left(\hat{x}_i - \epsilon u_k + \sum_{j \neq i} \left(x_j^* + \frac{\epsilon}{n-1} u_k \right) \right) = p^* \left(\hat{x}_i + \sum_{j \neq i} x_j^* \right)$$

from which $p^* x_i^* \leq p^* \bar{x}_i$.

ii. If $\hat{x}_i \succ_i x_i^*$ then $p^* \hat{x}_i > p^* x_i^*$:
Suppose that $\exists \hat{x}_i$ such that $\hat{x}_i \succ_i x_i^*$ and $p^* \hat{x}_i = p^* x_i^*$. Then by continuity of $\succsim_i$, $\exists \lambda < 1$ such that $\lambda \hat{x}_i \succ_i x_i^*$. From part i. above then $p^*(\lambda \hat{x}_i) \geq p^* x_i^*$. But if $p^* \hat{x}_i = p^* x_i^*$, then $\lambda p^* \hat{x}_i < p^* x_i^*$ since $\lambda < 1$. A contradiction. □

(c) UNFINISHED

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Pareto efficiency

QII.2 F2008

Consider an economy with N consumers and L goods $\mathcal{E} = (\mathbb{R}_+^L, \succsim_i, e_i, i = 1, \ldots, N)$. The preference of consumer $i = 1, \ldots, N$ can be represented by a differentiable function $u_i : \mathbb{R}_+^L \to \mathbb{R}$ that satisfies $Du_i(x_i) \gg 0$. Answer the following questions.

(a) If u_i satisfies an additional assumption, then $x^* \in A$, $x^* \gg 0$ (A is the set of feasible allocations) is Pareto efficient if and only if x^* solves the following maximization problem

$$\max_{x \in A} \sum_{i=1}^N a_i u_i(x_i)$$

for some $a \in \mathbb{R}_{++}^N$. What is this assumption?

(b) Define competitive equilibrium with transfer in this economy. Also derive a system of equations such that $(x^*, p^*) \gg 0$ is a competitive equilibrium with transfer if and only if $(x^*, p^*) \gg 0$ solves the system of equations for some multipliers (again state any additional assumption you used).

(c) Using the results from (a) and (b), prove the following version of the second welfare theorem in this economy (with an appropriate assumption on utility functions): "If $x^* \in \mathbb{R}_{++}^{LN}$ is Pareto efficient, then there exists a price vector $p^* \in \mathbb{R}_{++}^L$ such that (x^*, p^*) is a competitive equilibrium with transfer".

Solution

(a) **Claim 7.** *Consider economy $\mathcal{E} = (\mathbb{R}_+^L, \succsim_i, e_i, i = 1, \ldots, N)$. If the preference of consumer $i = 1, \ldots, N$ can be represented by a concave function u_i, then a feasible allocation x^* is Pareto efficient if and only if x^* solves*

$$\max_{x \in A} \sum_{i=1}^N a_i u_i(x_i)$$

where A is the set of feasible allocations.

(b)

Definition 17. (x^*, p^*) is a competitive equilibrium with transfers if $\exists w \in \mathbb{R}^{L\times N}$

i. x^* is feasible, i.e. $\sum_{i=1}^N x_i^* \leq \sum_{i=1}^n e_i$

ii. $x_i^* \in \text{argmax}_{x_i \in B(p^*, w_i)} u_i(x_i)$

iii. $p^* \sum_{i=1}^N w_i \leq p^* \sum_{i=1}^N e_i$

UNFINISHED

(c) UNFINISHED

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Core

QII.1 S2008

Consider a pure exchange economy with n consumers $i = 1, \ldots, N$, each having initial endowment $e_i \in \mathbb{R}_+^L$ and preferences $\succcurlyeq_i$ assumed to be complete preorders on $\mathbb{R}_+^L$

(a) Define the core of this economy.

(b) State and prove the Equal Treatment Property for the core of a pure exchange economy.

(c) State the Debreu-Scarf Theorem and discuss its economic significance.

Solution

(a) **Definition 18.** Let $\mathcal{E} = (I, (\mathbb{R}_+^L, \succcurlyeq_i, e_i)_{i\in\in I=\{1,\ldots,N\}})$ be a pure exchange economy. Its core is

$$core(\mathcal{E}) = \{x : x_i \in \mathbb{R}_+^L, \forall i \in I, \textstyle\sum_{i\in I} x_i \leq \sum_{i\in I} e_i,$$
$$\text{and } \nexists S \subseteq I \text{ for which } \exists \hat{x} \in \mathbb{R}_+^{LN}, \hat{x}_i \succ_i x_i \ \forall i \in S, \text{and } \textstyle\sum_{i\in S} \hat{x}_i \leq \sum_{i\in S} e_i\}$$

(b) **Theorem 12.** *Let $\mathcal{E} = (I, (\mathbb{R}_+^L, \succcurlyeq_i, e_i)_{i\in\{1,\ldots,N\}})$ be a pure exchange economy, where $\succcurlyeq_i$ is complete, strictly convex, strictly monotone and continuous preorder on $\mathbb{R}_+^L$ for each $i \in \{1, \ldots, N\}$. Let $\mathcal{E}^R = (NR, ((\mathbb{R}_+^L, \succcurlyeq_{ir}, e_{ir})_{i\in\{1,\ldots,N\}})_{r\in\{1,\ldots,R\}})$ be an R-fold replica economy of $\mathcal{E}$. If $x^* \in core(\mathcal{E}^R)$, then $x_{ir}^* = x_{is}^*$ for all $i \in \{1, \ldots, N\}$, all $r, s \in \{1, \ldots, R\}$.*

Proof. Suppose not, let $x^* \in core(\mathcal{E}^R)$, and (without loss of generality) assume that for $i = 1$, $r = 1$ we have $x_{11}^* \neq x_{is}^*$, and $x_{1s}^* \succcurlyeq_1 x_{11}^*$ for all $s \in \{2, \ldots, R\}$. Denote $\bar{x}_i = \frac{1}{R}\sum_{r=1}^R \bar{x}_{ir}^*$. Because $x_{1s}^* \succcurlyeq_1 x_{11}^*$ and $\succcurlyeq_1$ are strictly convex, $\bar{x}_1^* \succ_1 x_{11}^*$. Consider now a coalition $S = \{11, 21, \ldots, N1\}$. By continuity of $\succcurlyeq_1$, $\exists \epsilon > 0$ and $k \in \{1, \ldots, L\}$ such that $\hat{x}_{11} = \bar{x}_1 - \epsilon u_k \succ_1 x_{11}^*$, where u_k is the k-th unit basis vector in $\mathbb{R}^L$. By strict monotonicity of $\succcurlyeq_j$, we have $\hat{x}_{j1} = x_{j1}^* + \frac{\epsilon}{N-1}u_k = \bar{x}_j + \frac{\epsilon}{N-1}u_k \succ_j x_{j1}^*$ for all $j \in \{2, \ldots, N\}$. Moreover,

$$\sum_{i=1}^N \hat{x}_{i1} = \bar{x}_1 - \epsilon u_k + \sum_{i=2}^N \left(\bar{x}_j + \frac{\epsilon}{N-1}u_k\right)$$
$$= \sum_{i=1}^N \bar{x}_j = \sum_{i=1}^N \frac{1}{R}\sum_{r=1}^R x_{ir}^* = \frac{1}{R}\sum_{r=1}^R\sum_{i=1}^N x_{ir}^* = \frac{1}{R}\sum_{r=1}^R\sum_{i=1}^N e_{ir} = \sum_{i=1}^N e_{i1}$$

Thus allocation $\hat{x}$ is feasible in coalition S and is preferred to x^* by each member of S. Therefore S is a blocking coalition, which is a contradiction since $x^* \in core(\mathcal{E}^R)$. □

(c) **Theorem 13.** *Let* $\mathcal{E} = (I, (\mathbb{R}^L_+, \succcurlyeq_i, e_i)_{i\in\{1,\dots,N\}})$ *be a pure exchange economy, where* $\succcurlyeq_i$ *is complete, strictly convex, strictly monotone and continuous preorder on* $\mathbb{R}^L_+$ *for each* $i \in \{1,\dots,N\}$, *and* $\sum_{i=1}^N e_i \gg 0$. *Let* $(E^R)_{R=1}^\infty$ *be a sequence of replica economies. Then* $CE(\mathcal{E}) = \cap_{R=1}^\infty core(\mathcal{E}^R)$.

UNFINISHED.

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First welfare theorem
QII.2 S2008

Consider a pure exchange economy with N consumers ($i = 1,\dots,N$) and L commodities. Consumer i has an initial endowment vector $e_i \in \mathbb{R}^L_+$ and a rational preference $\succcurlyeq_i$ (complete, reflexive, and transitive binary relation, i.e. complete preorder) on the consumption set $X_i \subset \mathbb{R}^L_+$. You may assume that the preferences are continuous.

(a) Define competitive (Walrasian) equilibrium and Pareto efficient (optimal) allocation in this pure exchange economy.

(b) State and prove the first welfare theorem. If you need any extra assumption(s), make sure to state all of them formally and clearly.

(c) Present one example of pure exchange economy where the first welfare theorem fails to hold. Explain which assumption you used in (b) is violated.

Solution

(a) **Definition 19.** Let $\mathcal{E} = (I, (X_i, \succcurlyeq_i, e_i)_{i\in\{1,\dots,N\}})$ be a pure exchange economy. (x^*, p^*) is a competitive equilibrium

i. x^* is feasible, i.e. $\sum_{i=1}^N x_i^* \le \sum_{i=1}^N e_i$

ii. $x_i^* \in x_i^*(p^*, e_i)$ for all $i \in I$, where
$x_i^*(p^*, e_i) = \{x_i \in X_i : p^* x_i \le p^* e_i,\ x_i \succcurlyeq_i x_i', \forall x_i' \in X_i \text{ such that } p^* x_i' \le p^* e_i\}$

Definition 20. Let $\mathcal{E} = (I, (X_i, \succcurlyeq_i, e_i)_{i\in\{1,\dots,N\}})$ be a pure exchange economy. Allocation $x^* \in \mathbb{R}^{LN}_+$ is weakly Pareto optimal if

i. $x_i^* \in X_i$

ii. $\sum_{i=1}^N x_i^* \le \sum_{i=1}^N e_i$

iii. $\nexists \hat{x} \in \mathbb{R}^{LN}_+$ such that $\hat{x}_i \in X_i$, $\sum_{i=1}^N \hat{x}_i \le \sum_{i=1}^N e_i$ and $\hat{x}_i \succ_i x_i^*$ for all $i \in \{1,\dots,N\}$

Allocation $x^* \in \mathbb{R}^{LN}_+$ is strongly Pareto optimal if

i. $x_i^* \in X_i$

ii. $\sum_{i=1}^N x_i^* \le \sum_{i=1}^N e_i$

iii. $\nexists \hat{x} \in \mathbb{R}^{LN}_+$ such that $\hat{x}_i \in X_i$, $\sum_{i=1}^N \hat{x}_i \le \sum_{i=1}^N e_i$, $\hat{x}_i \succcurlyeq_i x_i^*$ for all $i \in \{1,\dots,N\}$, and $\hat{x}_i \succ_i x_i^*$ for at least one $i \in \{1,\dots,N\}$

(b) **Theorem 14.** *Consider a pure exchange economy* $\mathcal{E} = (I, (X_i, \succcurlyeq_i, e_i)_{i\in I})$ *where for all* $i \in I = \{1,\dots,N\}$ $\succcurlyeq_i$ *is a complete, locally nonsatiated preorder on* $X_i = \mathbb{R}^L_+$. *If* $x^* \in R^{LN}$ *is allocation in the competitive equilibrium* (x^*, p^*), *then* x^* *is strongly Pareto optimal.*

Proof. Suppose not, then there exists a feasible allocation $\hat{x} \in R^{LN}$ such that for all $i \in I$, $\hat{x}_i \succsim_i x_i^*$ and $\exists i' \in I$ such that $\hat{x}_{i'} \succ_{i'} x_{i'}^*$. Since $\succsim_i$ are locally nonsatiated we know that for all $i \in I$ it holds $p^* x_i^* = p^* e_i$ and also $p^* \hat{x}_i \geq p^* e_i$, $p^* \hat{x}_{i'} > p^* e_{i'}$. Taking the sum for across i we obtain $\sum_{i \in I} p^* \hat{x}_i > \sum_{i \in I} p^* e_i$ or $p^* \sum_{i \in I} \hat{x}_i > p^* \sum_{i \in I} e_i$. But since $\hat{x}$ is feasible $\sum_{i \in I} \hat{x}_i \leq \sum_{i \in I} e_i$ and so $p^* \sum_{i \in I} \hat{x}_i \leq p^* \sum_{i \in I} e_i$, which is a contradiction. □

(c) First welfare theorem may fail to hold if

i. $\succsim_i$ are locally satiated (e.g. two goods, two consumer economy with $X_i = \mathbb{R}_+^2$, endowments $e_1 = (1,1), e_2 = (1,1)$, and preferences $u_1(x) = 1$ for all $x \in X_1$, $u_2(x) = x_1 + x_2$ for all $x \in X_2$)

ii. N, L are infinite (e.g. OLG model without storage technology and money),

iii. X_i are not convex sets (e.g. two goods, two consumer economy with $X_i = \mathbf{N}_+^2$, endowments $e_1 = (1,0), e_2 = (0,1)$, and Cobb-Douglas preferences $u_i(x) = x_1^0.5 x_2^0.5$ for $i = 1, 2$)

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Aldo Rustichini

Linear games
QIII.1 S2009

An extensive form game (EFG) is said to be linear if every information set is crossed at most once by every history.

(a) Give an example of an EFG which is not linear.

(b) Compare linear games and games with perfect recall. Is one of the two a subset of the other? Prove your answer.

Solution

(a) UNFINISHED

(b) **Definition 21.** An extensive form game Γ has perfect recall if $\forall i \in I$, $\forall u \in U^i$, $\forall z \in Z$ it holds that $\#\{x \in X : x \in P(z) \cap u\} \leq 1$

Claim 8. *If Γ has a perfect recall, then Γ is linear.*

Proof. Suppose that Γ has a perfect recall, but is not linear. Then $\exists i \in I,\ u \in U_i,\ x_1, x_2 \in u$ such that $x_2 \succcurlyeq x_1$ and $x_2 \xrightarrow{a} x_1$ for some $a \in A^i$. Since Γ has a perfect recall $x_1, x_2 \in u$ implies that $\exists x_3$, such that $x_3 \succcurlyeq x_2$, $x_3 \xrightarrow{a} x_2$ and since $x_1, x_2 \in u$ also $x_3, x_2 \in u'$. But then $u' = u$, $x_1, x_2, x_3 \in u$, $x_3 \succcurlyeq x_2 \succcurlyeq x_1$ and $x_3 \xrightarrow{a} x_2 \xrightarrow{a} x_1$. We can continue this way to get $x_n \succcurlyeq \ldots \succcurlyeq x_2 \succcurlyeq x_1$, $x_n \xrightarrow{a} \ldots \xrightarrow{a} x_2 \xrightarrow{a} x_1$ for any n. Since Γ is finite with K nodes, for $n > K$ this implies that $\exists k$ such that $x_k \xrightarrow{a} x_{k-1} \ldots \xrightarrow{a} x_k$, $x_k \succcurlyeq x_{k-1} \succcurlyeq \ldots \succcurlyeq x_k$ which is a contradiction. □

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Correlated equilibria
QIII.1 F2008

(a) Define the correlated strategies and correlated equilibria;

(b) Prove that the set of correlated equilibria and the set correlated equilibrium payoffs are closed and convex.

(c) Give an example of a game where correlated equilibria and Nash Equilibria are the same.

(d) Give an example of a game where the set of correlated equilibria is not the convex hull of the set of Nash equilibria.

Solution

(a) **Definition 22.** Let $\Gamma = (I, (A_i)_{i \in I}, (u_i)_{i \in I})$, be a normal form game. The set of correlated strategies is $\Delta(A) = \{\mu \in \mathbb{R}_+^{\#A} : \sum_{k=1}^{\#A} \mu_k = 1\}$ where A is the set of pure strategy action profiles $A = \times_{i=1}^{n} A_i$.

Definition 23. A correlated equilibrium is a correlated strategy $\mu \in \Delta(A)$ such that $\forall i \in I$, $\forall a^i \in A^i$ with $\mu(a^i) > 0$ it holds that $\forall b^i \in A^i$

$$\sum_{a^{-i} \in A^{-i}} u_i(a^i, a^{-i})\mu(a^{-i}|a^i) \geq \sum_{a^{-i} \in A^{-i}} u_i(b^i, a^{-i})\mu(a^{-i}|a^i)$$

(b) **Claim 9.** *The set of correlated equilibria and the set correlated equilibrium payoffs are nonempty, closed and convex.*

Proof. Since any game has a mixed strategy NE equilibrium, and a mixed strategy NE equilibrium s defines a correlated equilibrium Pr_s the set of correlated equilibria is nonempty.

Using $\mu(a^{-i}|a^i) = \frac{\mu(a^{-i}, a^i)}{\mu(|a^i)}$ we can rewrite the definition of a correlated equilibria as a set of linear inequalities which μ has to satisfy

$$\sum_{a^{-i} \in A^{-i}} [u_i(a^i, a^{-i}) - u_i(b^i, a^{-i})]\mu(a^{-i}, a^i) \geq 0$$

for all $i \in I$, all $a^i, b^i \in A^i$. The subset of $\mathbb{R}^{\#A}$ given by the system of linear inequalities above is closed, and convex.

The set of correlated equilibrium payoffs are

$$CEP(G) = \{x \in \mathbb{R}^I : x = \textstyle\sum_{a \in A} u(a)\mu(a),\ \mu \in CE(G)\}$$

Since $\sum_{a \in A} u(a)\mu(a)$ is continuous in μ and $CE(G)$ is a nonempty, closed and convex set, also $CEP(G)$ is a nonempty, closed and convex set. $\square$

(c) UNFINISHED

(d) Based on Aumann (1974). Consider the following game of chicken, with two players whose action sets are $A^i = \{chicken\ out, dare\}$ and with payoffs

	C	D
C	2,2	0,3
D	3,0	-1,-1

There are three Nash equilibria $NE_1 = ((1,0),(0,1))$, $NE_2 = ((1,0),(0,1))$, $NE_3 = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.

If the probability distribution over action profiles is

	C	D
C	p_{11}	p_{12}
D	p_{21}	p_{22}

the set of correlated equilibria has to satisfy the following constraints

$$
\begin{aligned}
2\frac{p_{11}}{p_{11}+p_{12}} &\geq 3\frac{p_{11}}{p_{11}+p_{12}} - \frac{p_{12}}{p_{11}+p_{12}} \\
3\frac{p_{21}}{p_{21}+p_{22}} - \frac{p_{22}}{p_{21}+p_{22}} &\geq 2\frac{p_{21}}{p_{21}+p_{22}} \\
2\frac{p_{11}}{p_{11}+p_{21}} &\geq 3\frac{p_{11}}{p_{11}+p_{21}} - \frac{p_{21}}{p_{11}+p_{21}} \\
3\frac{p_{12}}{p_{12}+p_{22}} - \frac{p_{22}}{p_{12}+p_{22}} &\geq 2\frac{p_{12}}{p_{12}+p_{22}} \\
p_{11}+p_{12}+p_{21}+p_{22} &= 1 \\
p_{11}, p_{12}, p_{21}, p_{22} &\geq 0
\end{aligned}
$$

and the inequalities can be simplified as $p_{12} \geq p_{11}$, $p_{21} \geq p_{22}$, $p_{21} \geq p_{11}$, $p_{12} \geq p_{22}$.

Then for example $p_{11} = p_{12} = p_{21} = \frac{1}{3}$ satisfies there constraints and gives payoffs $(\frac{5}{3}, \frac{5}{3})$, which is outside of the convex hull of Nash equilibria.

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Never a best response

QIII.2 S2008

For a finite normal form game:

(a) Define the following property: an action is "never a best response".

(b) Prove that an action is never a best response if and only if it not strictly dominated by a mixed strategy.

Solution

(a) **Definition 24.** A strategy $a^i \in A^i$ is never a best response, if
$\nexists \mu \in \Delta(A^{-i})\ \forall b^i \in A^i$, $u_i(a^i, \mu) \geq u_i(b^i, \mu)$, or alternatively if
$\forall \mu \in \Delta(A^{-i})\ \exists b^i \in A^i$, $u_i(b^i, \mu) > u_i(a^i, \mu)$

(b) **Claim 10.** *An action is never a best response if and only if it is strictly dominated by a mixed strategy.*

Proof.

$\Rightarrow$ We will prove the contraposition. Let $\hat{a}^i \in A^i$ be an action which is not strictly dominated and introduce the following notation

$$
U = \begin{pmatrix} u^i(a^1, a^{-i}_1) & \dots & u^i(a^i_{\#A^i}, a^{-i}_1) \\ \vdots & & \\ u^i(a^1, a^{-i}_{\#A^{-i}}) & \dots & u^i(a^i_{\#A^i}, a^{-i}_{\#A^{-i}}) \end{pmatrix} \qquad u = \begin{pmatrix} u^i(\hat{a}^i, a^{-i}_1) \\ \vdots \\ u^i(\hat{a}^i, a^{-i}_{\#A^{-i}}) \end{pmatrix}
$$

Since $\hat{a}^i$ is not strictly dominated $\nexists x \in \mathbb{R}^n$, $n = \#A^i$ such that

$$
\begin{aligned}
Ux &\gg u \\
I_n x &\geq 0_n \\
e_n^T x &= 1
\end{aligned}
$$

By the theorem of alternative this is equivalent to $\exists \mu \in \mathbb{R}^m, \lambda \in \mathbb{R}^n, \nu \in \mathbb{R}$, $\mu \geq 0, \lambda \geq 0$, where $m = \#A^{-i}$, $n = \#A^i$, such that

$$\mu^T U + \lambda^T I_n + \nu e_n^T = 0$$
$$\mu^T u + \lambda^T 0_n + \nu 1 \geq 0$$
$$\mu^T (u + e_n) + \lambda^T 0_n + \nu 1 > 0$$

or

$$\mu^T U = -(\lambda^T + \nu e_n^T)$$
$$\mu^T u \geq -\nu$$
$$\mu^T (u + e_n) > -\nu$$

Since $\lambda \geq 0$ from the first two we get

$$\mu^T u e_n^T \geq -\nu e_n^T \geq -(\lambda^T + \nu e_n^T) = \mu^T U \qquad \Rightarrow \qquad \mu^T u e_n^T \geq \mu^T U$$

The choice of (λ, μ, ν) is not unique, i.e. if (λ, μ, ν) satisfy the above constraints then so do $(a\lambda, a\mu, a\nu)$ for any $a > 0$. Thus we can normalize so that $\mu^T e_m = 1$. But then μ can be interpreted as a belief on A^{-i}, and $\mu^T u e_n^T \geq \mu^T U$ implies that this belief rationalizes the choice of action $\hat{a}^i$, and so $\hat{a}^i$ is not never a best response. By contraposition, if $a^i \in A^i$ is never a best response then a^i is strictly dominated.

$\Leftarrow$ Suppose $a^i \in A^i$ is strictly dominated by a mixed strategy $s^i \in S^i$, i.e. $\forall a^{-i} \in A^{-i}$, $u^i(a^i, a^{-i}) < u^i(s^i, a^{-i})$. Then for any belief $\mu \in \Delta(A^{-i})$ since necessarily $\mu(a^{-i}) > 0$ for some $a^{-i} \in A^{-i}$

$$\sum_{a^{-i} \in A^{-i}} u^i(a^i, a^{-i})\mu(a^{-i}) < \sum_{a^{-i} \in A^{-i}} u^i(s^i, a^{-i})\mu(a^{-i})$$

It must be also that $\exists b^i \in \mathit{support}(s^i) \subseteq A^i$ such that

$$\sum_{a^{-i} \in A^{-i}} u^i(s^i, a^{-i})\mu(a^{-i}) \leq \sum_{a^{-i} \in A^{-i}} u^i(b^i, a^{-i})\mu(a^{-i})$$

and so $\forall \mu \in \Delta(A^{-i})$ $\exists b^i \in A^i$ such that

$$u^i(a^i, \mu) = \sum_{a^{-i} \in A^{-i}} u^i(a^i, a^{-i})\mu(a^{-i}) < \sum_{a^{-i} \in A^{-i}} u^i(b^i, a^{-i})\mu(a^{-i}) = u^i(b^i, \mu)$$

which means that a^i is never a best response. □

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Nash equilibrium

QIII.1 F2007

(a) Show that every finite game possesses a Nash equilibrium in which no player places a strictly positive probability on a weakly dominated strategy.

(b) Improve this result from (a) by showing that every finite game possesses a Nash equilibrium σ in which for every player i, σ_i is not dominated.

(c) Show by an example that the result in the previous point (b) requires finiteness.

Solution

(a) *Proof.*

Step 1. Show that $PE(G) \neq \emptyset$ and $PE(G) \subseteq NE(G)$.

Consider (ϵ_n), a sequence of perturbations. For every ϵ_n, $NE(G^{\epsilon_n}) = \{\hat{s} \in S_{\epsilon_n} : \hat{s}_n \in BR_{\epsilon_n}(\hat{s}_n)\}$ where $BR_{\epsilon_n}(\hat{s}_n) = \times_{i=1}^{I} BR^i_{\epsilon_n}(\hat{s}_n)$, $BR^i_{\epsilon_n}(\hat{s}_n) = \{s^i \in S^i_{\epsilon_n} : \forall t^i \in S^i_{\epsilon_n}\ u_i(s^i, \hat{s}_n^{-i}) \geq u_i(t^i, \hat{s}_n^{-i})\}$ and $S^i_{\epsilon_n} = \{s^i \in \mathbf{R}_+^{\#A_i} : s^i_k \geq \epsilon_n{}^i_k,\ \sum_{k=1}^{\#A_i} s^i_k = 1\}$. Using Kakutani's fixed point theorem we can show that $NE(G^{\epsilon_n}) \neq \emptyset$ in the same way the existence of $NE(G)$ was proved.

Since S is compact and $\hat{s}_n \in S$ for all n, there exists a convergent subsequence $\hat{s}_{n_k} \to \hat{s}$ with $\hat{s} \in S$. Thus $\hat{s} \in PE(G)$ and $PE(G) \neq \emptyset$. For all n_k, all $i \in I$, all $t^i \in A^i$, for $\hat{s}_{n_l} \in NE(G^{\epsilon_{n_k}})$ it holds

$$u_i(\hat{s}^i_{n_k}, \hat{s}^{-i}_{n_k}) \geq u_i(t^i, \hat{s}^{-i}_{n_k})$$

Taking the limit $k \to \infty$ on both sides and using the fact that

$$u_i(s^i, s^{-i}) = \sum_{a^i \in A^i} \sum_{a^{-i} \in A^{-i}} s^i(a^i) \prod_{j \neq i} s^j(a^j) u_i(a^i, a^{-i})$$

is continuous in (s^i, s^{-i}) we have $u_i(\hat{s}^i, \hat{s}^{-i}) \geq u_i(t^i, \hat{s}^{-i})$. Thus $\hat{s} \in NE(G)$.

Step 2. Show that weakly dominated strategies are not played in $PE(G)$

Let $\hat{s} \in PE(G)$ and suppose that $\exists i \in I$, $a^i_k \in A^i$ such that $\hat{s}^i(a^i_k) > 0$, and $\exists t^i \in S^i$ for which $\forall a^{-i} \in A^{-i}$, $u^i(a^i_k, a^{-i}) \leq u^i(t^i, a^{-i})$ and $\exists b^{-i} \in A^{-i}$, $u^i(a^i_k, b^{-i}) < u^i(t^i, b^{-i})$.

Since $\hat{s}^i(a^i_k) > 0$, $s^i_n(a^i_k) \to \hat{s}^i(a^i_k)$, $\epsilon^i_n(a^i_k) \to 0$ there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ it holds that $s^i_n(a^i_k) > \epsilon^i_n(a^i_k)$. Let $\gamma_n = s^i_n(a^i_k) - \epsilon^i_n(a^i_k)$.

Construct a mixed strategy $\sigma^i_n \in S^i_{\epsilon_n}$ as

$$\sigma^i_n(a^i_j) = \begin{cases} s^i_n(a^i_k) + \gamma_n t^i(a^i_j) & j \neq k \\ \epsilon^i_n(a^i_j) + \gamma_n t^i(a^i_j) & j = k \end{cases}$$

Then

$$\begin{aligned}
&u(\sigma^i_n, s^{-i}_n) - u(s^i_n, s^{-i}_n) \\
&= \sum_{a^{-i} \in A^{-i}} \sum_{a^i \in A^i} \prod_{j \neq i} s^j_n(a^j) \sigma^i_n(a^i) u^i(a^i_j, a^{-i}) - \sum_{a^{-i} \in A^{-i}} \sum_{a^i \in A^i} \prod_{j \neq i} s^j_n(a^j) s^i_n(a^i) u^i(a^i, a^{-i}) \\
&= \sum_{a^{-i} \in A^{-i}} \sum_{a^i \in A^i} \prod_{j \neq i} s^j_n(a^j) [\sigma^i_n(a^i) - s^i_n(a^i)] u^i(a^i, a^{-i}) \\
&= \sum_{a^{-i} \in A^{-i}} \prod_{j \neq i} s^j_n(a^j) \left(\sum_{a^i \in A^i, a^i \neq a^i_k} \gamma_n t^i(a^i) u^i(a^i, a^{-i}) + (\epsilon^i_n(a^i_k) + \gamma_n t^i(a^i_k) - s^i_n(a^i_k)) u^i(a^i_k, a^{-i}) \right) \\
&= \sum_{a^{-i} \in A^{-i}} \prod_{j \neq i} s^j_n(a^j) \left(\sum_{a^i \in A^i, a^i \neq a^i_k} \gamma_n t^i(a^i) u^i(a^i, a^{-i}) + \gamma_n (t^i(a^i_k) - 1) u^i(a^i_k, a^{-i}) \right) \\
&= \gamma_n \sum_{a^{-i} \in A^{-i}} \prod_{j \neq i} s^j_n(a^j) \left(\sum_{a^i \in A^i} t^i(a^i) u^i(a^i, a^{-i}) - u^i(a^i_k, a^{-i}) \right) \\
&= \gamma_n \sum_{a^{-i} \in A^{-i}} \prod_{j \neq i} s^j_n(a^j) \left(u^i(t^i, a^{-i}) - u^i(a^i_k, a^{-i}) \right)
\end{aligned}$$

Since $s_n^j(a^j) \geq \epsilon_n^j(a^j) > 0$, $\gamma_n > 0$, $u^i(t^i, a^{-i}) \geq u^i(a_k^i, a^{-i})$ for all $a^{-i} \in A^{-i}$, and $u^i(t^i, b^{-i}) > u^i(a_k^i, b^{-i})$ for some $b^{-i} \in A^{-i}$ we have

$$u(\sigma_n^i, s_n^{-i}) - u(s_n^i, s_n^{-i}) > 0$$

But this is a contradiction with $s_n^i \in BR^i(s_n)$. Hence it must be that if a_k^i is weakly dominated then $\hat{s}^i(a_k^i) = 0$. □

(b) **Claim 11.** *If σ^* is a perfect equilibrium, then for every i, σ_i^* is not dominated.*

Proof. Suppose not. Then $\exists i \in I, t^i \in S^i$, such that

$$\forall a^{-i} \in A^{-i},\ u^i(\sigma_i^*, a^{-i}) \leq u^i(t_i, a^{-i}) \quad (1)$$
$$\exists b^{-i} \in A^{-i},\ u^i(\sigma_i^*, b^{-i}) < u^i(t_i, b^{-i}) \quad (2)$$

Because σ^* is a perfect equilibrium, there exists a sequence of completely mixed strategy profiles σ_n, $\sigma_n \to \sigma^*$ and $\exists N$ such that for all $n > N$, for all $i \in I$, $\sigma_i^* \in BR^i(\sigma_n)$. Since σ_n are fully mixed $Pr_{\sigma_n}(a^{-i}) > 0$ for all $a^{-i} \in A^{-i}$. Thus by multiplying (1)-(2) with $Pr_{\sigma_n}(a^{-i})$ and summing across A^{-i} we get

$$u^i(\sigma_i^*, \sigma_n) = \sum_{a^{-i} \in A^{-i}} u^i(\sigma_i^*, a^{-i}) Pr_{\sigma_n}(a^{-i}) < \sum_{a^{-i} \in A^{-i}} u^i(t_i, a^{-i}) Pr_{\sigma_n}(a^{-i}) = u^i(t_i, \sigma_n)$$

which is a contradiction since then $\sigma_i^* \notin BR^i(\sigma_n)$ for any n. □

(c) UNFINISHED

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Correlated equilibria

QIII.2 F2007

(a) Define the set of correlated strategies and correlated equilibria for a finite game.

(b) Define an augmented game, and show how the Bayesian-Nash equilibrium of the augmented game and the correlated equilibrium are related.

(c) Prove that the set of correlated equilibrium payoffs is a closed, convex, non-empty set.

Solution

(a) Same as in QIII.1 F2008 part (a).

(b) UNFINISHED

(c) Same as in QIII.1 F2008 part (b).

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Extensive form games

QIII.1 S2007

Consider extensive form games that are finite (that is, that have a finite set of nodes).

(a) Define an extensive form linear game.

(b) Prove that for any linear game, any player in the game, and any behavioral strategy of the player there is a mixed strategy of the same player that induces the same probability distribution on final nodes for any pure strategy of the other players.

(c) Give an example to show that in a linear game for a mixed strategy of the player there may be no behavioral strategy that induces the same distribution on final nodes for some pure strategy of the other players.

Solution

(a) **Definition 25.** An extensive form game is $\Gamma = (I, X, \succcurlyeq, \mathcal{O}, Z, P, U, C, u)$ where I is the set of players, X is the set of nodes which is partially ordered by $\succcurlyeq$, $\mathcal{O}$ is the initial node, Z is the set of final nodes, $P = (P^i)_{i\in I}$ is the partition of $X \setminus \{\mathcal{O}, Z\}$ into nodes controlled by players, $U = (U^i)_{i\in I}$, where $U^i = (U^i_j, \dots, U^i_k)$ is the partition of sets P^i into information sets, $C = (C_u)_{u\in U}$ is the choice partition, and $u : Z \to \mathbb{R}^I$ is the assignment of payoffs on final nodes. Partial order $\succcurlyeq$ is reflexive, antisymmetric, transitive, and satisfies: for all $x \in X$, $\mathcal{O} \succcurlyeq x$, for all $z \in Z$, $\nexists x \in X$, $x \neq z$, $x \succcurlyeq z$.

Definition 26. For any $x \in X$ denote $h(x) = \{y \in X : y \succcurlyeq x\}$. An extensive form game Γ is linear if $\#\{h(z) \cap u\} \leq 1$, for all $z \in Z$, all $u \in U$.

(b) **Claim 12.** *For any linear game G, for any player $i \in I$, and any behavioral strategy $b^i \in B^i$ of the player there is a mixed strategy $\sigma_i \in \Sigma_i$ that induces the same probability distribution on final nodes Z for any pure strategy $s^{-i} \in S^{-i}$ of the other players.*

Proof. Let $b^i \in B^i = \bigtimes_{u^i_k \in U^i} \Delta(C_{u^i_k})$ be the behavioral strategy of some player $i \in I$ in a linear game G. Define $\sigma^i(s^i) = \prod_{u^i_k \in U^i} b^i(u^i_k, s^i_k)$. Then $\sigma^i \in \Sigma^i = \Delta(S^i) = \Delta(\bigtimes_{u^i_k \in U^i} C_{u^i_k})$

i. for all $s^i \in S^i$, $\sigma^i(s^i) \geq 0$ since $b^i(u^i_k, s^i_k) \geq 0$

ii. $\sum_{s^i\in S^i} \sigma^i(s^i) = \sum_{s^i\in S^i} \prod_{u^i_k\in U^i} b^i(u^i_k, s^i_k) = 1$

and we need to show that b^i and σ^i induce the same probability distribution on Z for any $s^{-i} \in S^{-i}$. For any $z \in Z$, $h(z) = \{x \in X : x \succcurlyeq z\}$ defines the actions that lead to z. Denote this set of actions $a = (a^i, a^{-i})$, $a^i \in S^i$, $a^{-i} \in S^{-i}$.

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(c) UNFINISHED

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Perfect equilibria

QIII.1 F2006

Consider the set of perfect equilibria of a normal form game.

(a) Prove: a perfect equilibrium is a Nash equilibrium.

(b) Prove or disprove: the set of perfect equilibria is a closed set.

Solution

(a) **Definition 27.** A mixed strategy profile $\hat{s}$ is a perfect equilibrium, if there exists a sequence of perturbations (ϵ_n)[1] and sequence of mixed strategy profiles $(\hat{s}_n)$, such that $\epsilon_n(a^i) \to 0$ for all $i \in I$, all $a^i \in A^i$, $\hat{s}_n \to \hat{s}$, and for each n, $\hat{s}_n \in NE(G^{\epsilon_n})$ where G^{ϵ_n} is a game obtained from G by restricting each player i to the set of mixed strategies $S^i_{\epsilon_n} = \{s^i \in S^i : s^i(a^i) \geq \epsilon^i_n(a^i),\ \forall a^i \in A^i\}$.

Claim 13. *Every perfect equilibrium is a Nash equilibrium, $PE(G) \subseteq NE(G)$.*

Proof. Let $\hat{s}$ be a perfect equilibrium, then $\exists(\epsilon_n), (\hat{s}_n)$, such that $\epsilon_n(a^i) \to 0$ for all $i \in I$, all $a^i \in A^i$, $\hat{s}_n \to \hat{s}$, and for each n, $\hat{s}_n \in NE(G^{\epsilon_n})$. Thus $\forall n$, $\forall i \in I$, $\forall t^i \in S^i_{\epsilon_n}$

$$u_i(\hat{s}^i_n, \hat{s}^{-i}_n) \geq u_i(t^i, \hat{s}^{-i}_n)$$

Since S^i is compact and $\epsilon_n(a^i) \to 0$, for any $t^i \in S^i$ we can find a sequence (t^i_n), such that $t^i_n \in S^i_{\epsilon_n}$, and $t^i_n \to t^i$. Then $\forall n$, $\forall i \in I$, $\forall t^i_n \in S^i_{\epsilon_n}$

$$u_i(\hat{s}^i_n, \hat{s}^{-i}_n) \geq u_i(t^i_n, \hat{s}^{-i}_n)$$

Taking the limit as $n \to \infty$, and using the fact that u_i is continuous we get $\forall i \in I$, $\forall t^i \in S^i$

$$u_i(\hat{s}^i, \hat{s}^{-i}) \geq u_i(t^i, \hat{s}^{-i})$$

which means that $\hat{s} \in NE(G)$. □

(b) **Claim 14.** *The set of perfect equilibria $PE(G)$ is closed.*

Proof. Consider a sequence of perfect equilibria, $\hat{s}_n \in PE(G)$ for all n and $\hat{s}_n \to \hat{s}$. Then $\exists(\epsilon^m_n)_{m\in\mathbb{N}}, (\hat{s}^m_n)_{m\in\mathbb{N}}$, such that as $m \to \infty$, $\epsilon^m_n(a^i) \to 0$ for $\forall i \in I$, $\forall a^i \in A^i$, $\forall n$ and $\hat{s}^m_n \to \hat{s}_n$.

Since as $m \to \infty$, $\epsilon^m_n(a^i) \to 0$ we have $\epsilon^n_n(a^i) \to 0$ as $n \to \infty$.

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Nash equilibrium

QIII.1 F2005

Let $G = (S^1, \ldots, S^I, u^1, \ldots, u^I)$ be a normal form game.

(a) State (but do not prove) Kakutani's fixed point theorem.

(b) State the definition of a Nash equilibrium.

(c) Taking Kakutani's theorem as given, prove that G has a Nash equilibrium.

Solution

(a)

[1] i.e. $\epsilon_n = ((\epsilon_n(a^i))_{a^i \in A^i})_{i \in I}$, $\epsilon_n(a^i) \geq 0$, for all $i \in I$, all $a^i \in A^i$, $\sum_{a^i \in A^i} \epsilon_n(a^i) < 1$ for all $i \in I$

Theorem 15. *Let $S \subset R^m$ be nonempty, convex, compact set and let $F : S \to S$ be a nonempty valued, convex valued correspondence which has a closed graph. Then $\exists s \in S$ such that $s \in F(S)$.*

(b) **Definition 28.** Let $G = (S^1, \ldots, S^I, u^1, \ldots, u^I)$ be a normal form game. A Nash equilibrium is a strategy profile $\hat{s} \in S = \bigtimes_{i \in I} S^i = \bigtimes_{i \in I} \Delta(A^i)$ such that $\forall i \in I, \forall t^i \in S^i$ it holds that $u^i(\hat{s}^i, \hat{s}^{-i}) \geq u^i(t^i, \hat{s}^{-i})$.

(c) **Claim 15.** *Every normal form game G has at least one Nash equilibrium in mixed strategies.*

Proof. The proof is based on Kakutani's fixed point theorem. For any $s \in S$ denote

$$BR^i(s) = \{t^i \in S_i : \forall t^{i'} \in S^i \;\; u^i(t^i, s^{-i}) \geq u^i(t^{i'}, s^{-i})\}$$

and let $BR(S) = \bigtimes_{i \in I} BR^i(s)$. Then $BR^i : S \to S^i$, $BR : S \to S$, and we will show that BR has a fixed point.

Set $S \subset \mathbb{R}^{\sum_{i \in I} \# A_i}$ is nonempty, convex a compact since it is a product of I nonempty, convex a compact sets $S^i = \Delta(A^i) \subset \mathbb{R}^{\# A_i}$.

Correspondence BR^i is

i. nonempty valued:
S^i is a compact set and $u^i(\cdot, s^{-i})$ is linear and thus continuous on S^i. Therefore $BR^i(s) = \text{argmax}_{t^i \in S^i} u_i(t^i, s^{-i})$ is well defined.

ii. convex valued:
Fix $s \in S$, $i \in I$ and let $\bar{t}^i, \bar{\bar{t}}^i \in BR^i(s)$. Then for all $t^i \in S^i$

$$\sum_{a^i \in A^i} \bar{\bar{t}}^i(a^i) u^i(a^i, s^{-i}) = \sum_{a^i \in A^i} \bar{t}^i(a^i) u^i(a^i, s^{-i}) \geq \sum_{a^i \in A^i} t^i(a^i) u^i(a^i, s^{-i})$$

and so for any $\lambda \in (0, 1)$

$$\lambda \left(\sum_{a^i \in A^i} \bar{\bar{t}}^i(a^i) u^i(a^i, s^{-i}) \right) + (1 - \lambda) \left(\sum_{a^i \in A^i} \bar{t}^i(a^i) u^i(a^i, s^{-i}) \right)$$
$$\geq \lambda \sum_{a^i \in A^i} t^i(a^i) u^i(a^i, s^{-i}) + (1 - \lambda) \sum_{a^i \in A^i} t^i(a^i) u^i(a^i, s^{-i})$$

or

$$\sum_{a^i \in A^i} (\lambda \bar{\bar{t}}^i(a^i) + (1 - \lambda) \bar{t}^i(a^i)) u^i(a^i, s^{-i}) \geq \sum_{a^i \in A^i} t^i(a^i) u^i(a^i, s^{-i})$$

which means that $\lambda \bar{\bar{t}}^i + (1 - \lambda) \bar{t}^i \in BR^i(s)$.

iii. $BR^i(s)$ has a closed graph
Consider a sequence (s_n), $s_n \in S$, $s_n \to s$, $s \in S$ and let (t_n^i) be a sequence such that $t_n^i \in S^i$, $t_n^i \in BR^i(s_n)$ and $t_n^i \to t^i$, $t^i \in S^i$. We want to show that $t^i \in BR^i(s)$.
By definition, for all $t^{i'} \in S^i$

$$u^i(t_n^i, s_n^{-i}) \geq u^i(t^{i'}, s_n^{-i})$$

taking the limit on both sides and using the fact that $u(\cdot, \cdot)$ is continuous we have that for all $t^{i'} \in S^i$

$$u^i(t^i, s^{-i}) = u^i(\lim_{n \to \infty} t_n^i, \lim_{n \to \infty} s_n^{-i}) = \lim_{n \to \infty} u^i(t_n^i, s_n^{-i})$$
$$\geq \lim_{n \to \infty} u^i(t^{i'}, s_n^{-i}) = u^i(t^{i'}, \lim_{n \to \infty} s_n^{-i}) = u^i(t^{i'}, s^{-i})$$

which means that $t^i \in BR^i(s)$.

Since BR is a product of BR^i it is also nonempty valued, convex valued and has a closed graph. All assumptions of Kakutani's fixed point theorem are satisfied for $BR : S \to S$, and so $\exists \hat{s} \in S$ such that $\hat{s} \in BR(\hat{s})$. By definition of BR then $\forall i \in I$, $\forall t^i \in S^i$ it holds that $u^i(\hat{s}^i, \hat{s}^{-i}) \geq u^i(t^i, \hat{s}^{-i})$. Thus $\hat{s}$ is a Nash equilibrium. □

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References

Aumann, R. J. (1966). Existence of competitive equilibria in markets with a continuum of traders. *Econometrica*, 34(1):1–17.

Aumann, R. J. (1974). Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1(1):67–96.

Hildenbrand, W. (1969). Pareto optimality for a measure space of economic agents. *International Economic Review*, 10(3):363–372.