EUI Lectures in Quantitative Macroeconomics

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Materials at ftp.mpls.frb.fed.us/pub/research/mcgrattan

Outline of Lectures

I. Business cycle accounting: Methods and misunderstandings

II. Beyond business cycle accounting: Some applications

III. Back to methods: Nonlinearities and large state spaces

I. BCA: Methods and Misunderstandings

Some Background

- Want preliminary data analysis technique
- Goals:
  - Isolate promising classes of models/theories/stories
  - Guide development of theory
  - Avoid critiques of structural VARs

Idea of Approach

- Equivalence results:
  - Detailed models with frictions observationally equivalent to
  - Prototype growth model with time-varying wedges
- Accounting procedure:
  - Use theory plus data to measure wedges
  - Estimate stochastic process governing expectations
  - Feed wedges back one at a time and in combinations
  - How much of output, investment, labor accounted for by each?

Prototype Growth Model

- Consumption (c), labor (l), investment (x) solve

  $$\max_{\{c_t, l_t, x_t\}} E \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

  subject to

  $$c_t + (1 + \tau_{xt}) x_t \leq (1 - \tau_{lt}) w_t l_t + r_t k_t + T_t$$

  $$k_{t+1} = (1 - \delta) k_t + x_t$$

- Production: $y_t = A_t F(k_t, \gamma^t l_t)$

- Resource: $c_t + g_t + x_t = y_t$

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- Resource: $c_t + g_t + x_t = y_t$

Note: Time-varying wedges (in red) are <u>not</u> structural

First-order Conditions of Prototype Model

- Efficiency wedge:

$$y_t = A_t F(k_t, \gamma^t l_t)$$

- Labor wedge:

$$-\frac{U_{lt}}{U_{ct}} = (1 - \tau_{lt})(1 - \alpha) y_t / l_t$$

- Investment wedge:

$$(1 + \tau_{xt}) U_{ct} = \beta E_t U_{ct+1} \left[\alpha y_{t+1} / k_{t+1} + (1 + \tau_{xt+1})(1 - \delta) \right]$$

- Government consumption wedge:

$$c_t + g_t + x_t = y_t$$

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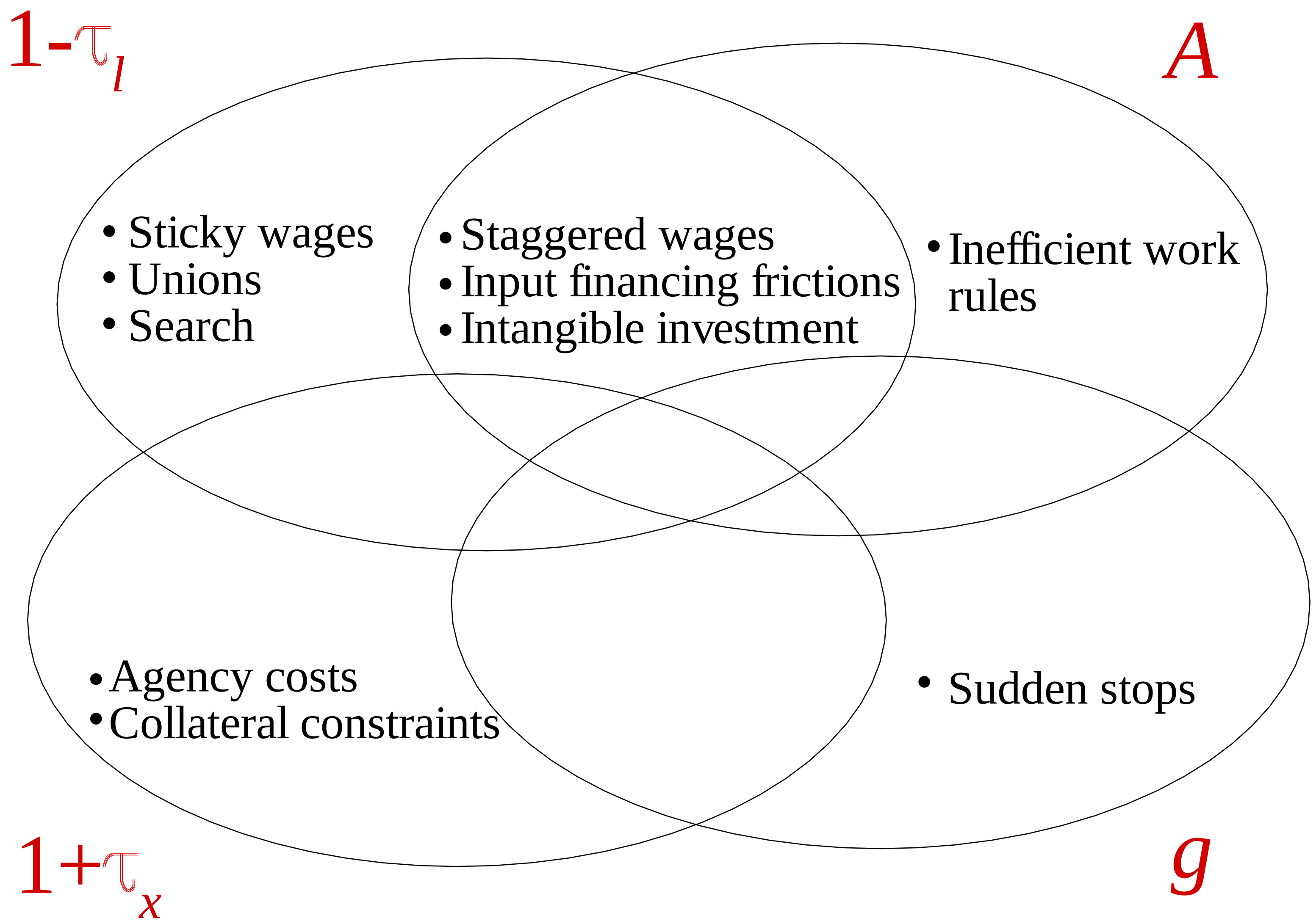
- Government consumption wedge:

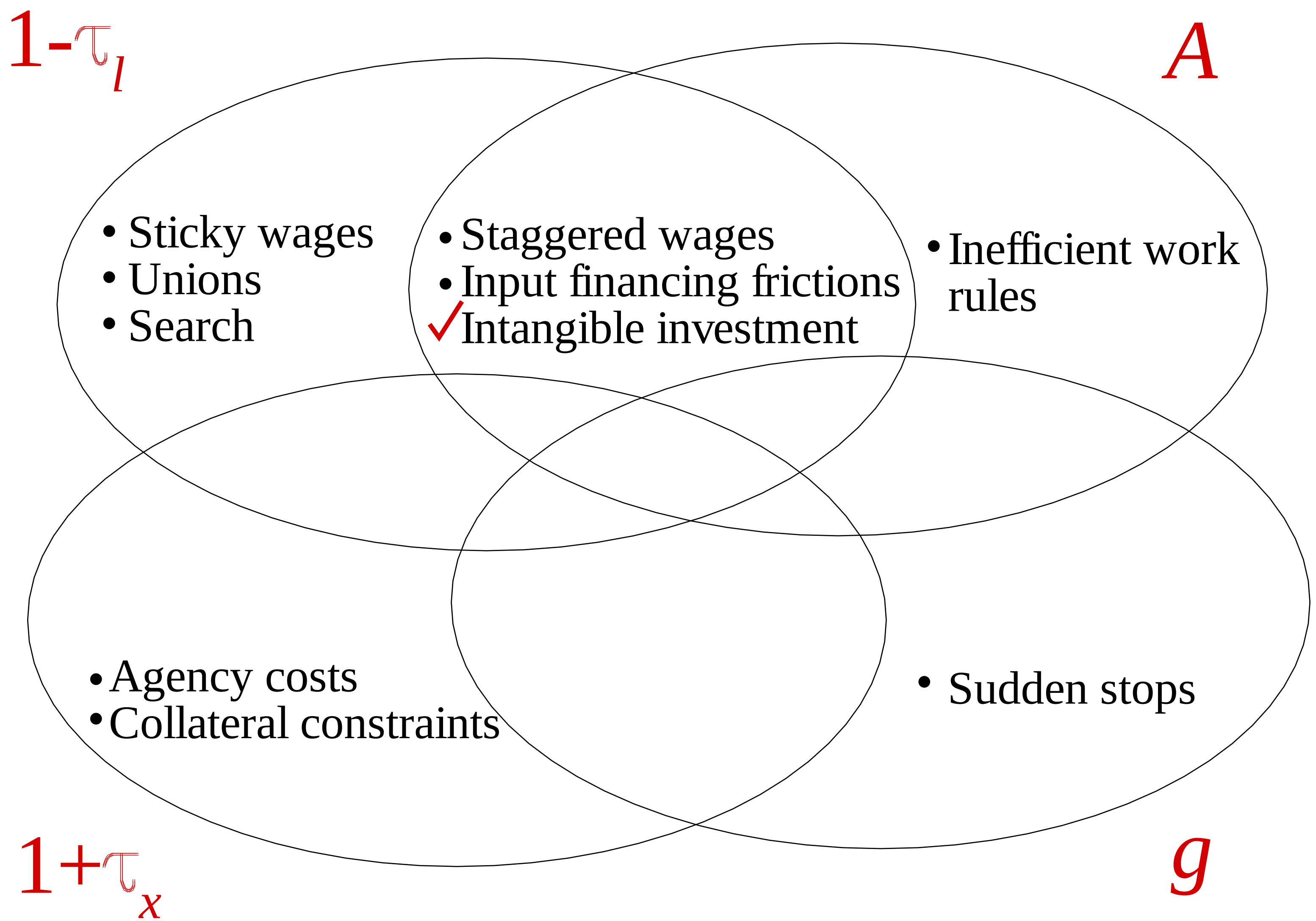
$$c_t + g_t + x_t = y_t$$

Next, consider mappings between this and other models

Mapping Between Original and Prototype Models



Mapping Between Original and Prototype Models



- Detailed economy of McGrattan-Prescott ("data"):
  - Two technologies for producing
    - Final goods and services
    - New trademarks and patents
  - Shocks to both productivities

- Prototype economy:
  - To account for "data," CKM need variation in A, τ_l
    - A variation is (partly) intangible capital movements
    - τ_l variation is due to mismeasuring productivity

Models with Intangible Investment

- Detailed economy of McGrattan-Prescott ("data"):
  - Two technologies for producing
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- Prototype economy:
  - To account for "data," CKM need variation in A, τ_l
    - A variation is (partly) intangible capital movements
    - τ_l variation is due to mismeasuring productivity

We'll return to this later...

Measuring Wedges

- Stochastic Process for wedges $s_t = [\log A_t, \tau_{lt}, \tau_{xt}, \log g_t]$
  - $s_{t+1} = P_0 + P s_t + Q\eta_{t+1}$
- Preferences and technology
  - $U(c, l) = \log c + \psi \log(1 - l)$
  - $F(k, l) = Ak^{\theta} l^{1-\theta}$
- With data from national accounts
  - Fix parameters of technology and preferences
  - Compute MLE estimates of P_0, P, Q

Recovering Wedges

- Model decision rules are $y(s_t, k_t), x(s_t, k_t), l(s_t, k_t)$

- Set:
  - $y(s_t, k_t) = y_t^{DATA}$
  - $x(s_t, k_t) = x_t^{DATA}$
  - $l(s_t, k_t) = l_t^{DATA}$
  - $g(s_t, k_t) = g_t^{DATA}$

  with k_t defined recursively from accumulation equation

- Solve for values of $s_t = [\log A_t, \tau_{lt}, \tau_{xt}, \log g_t]$

- Inputting these values gives exactly same series as in data

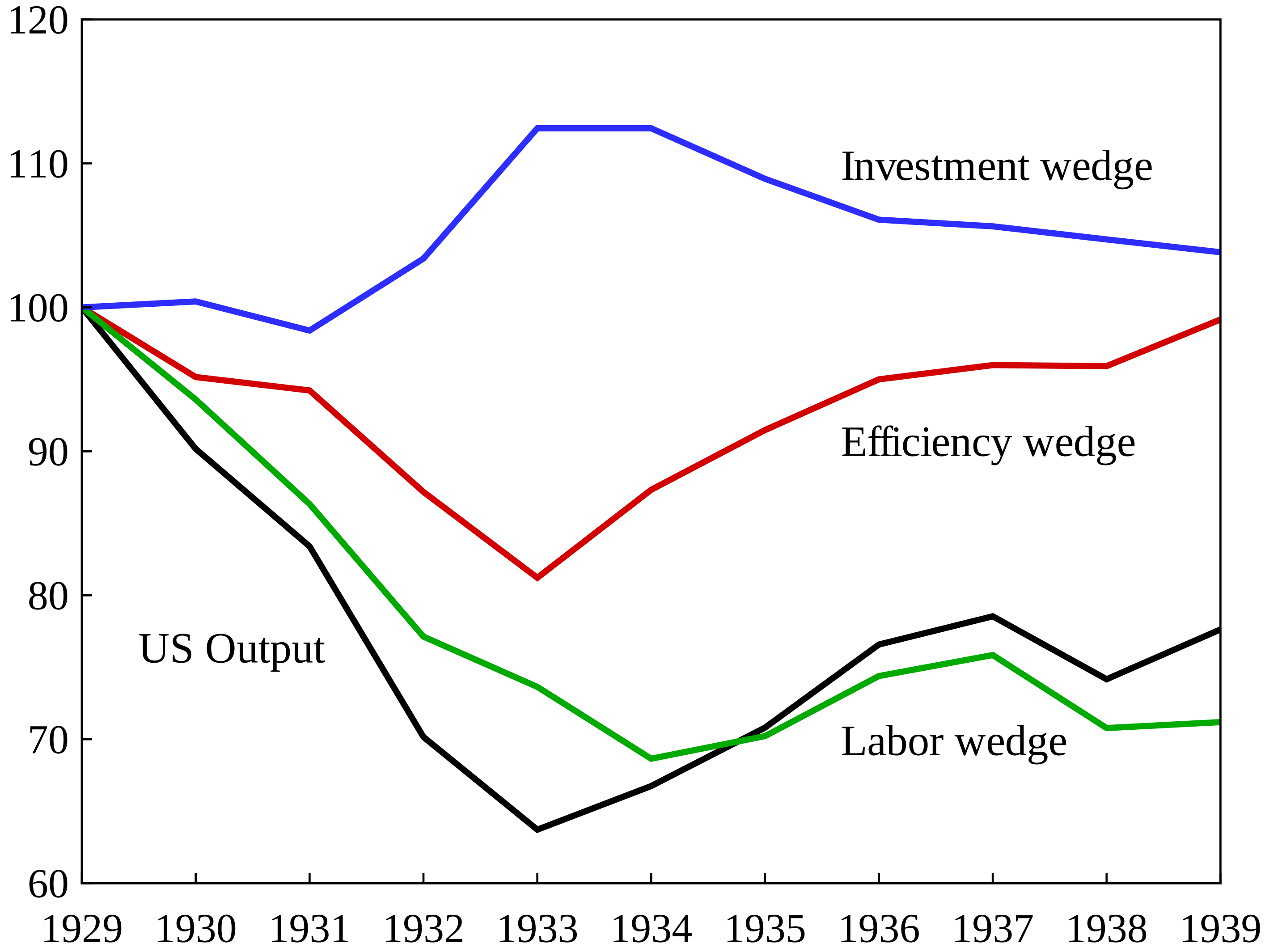
Main result for US Episodes

- Investment wedge plays small role in
  - Great Depression
  - Post WWII business cycles

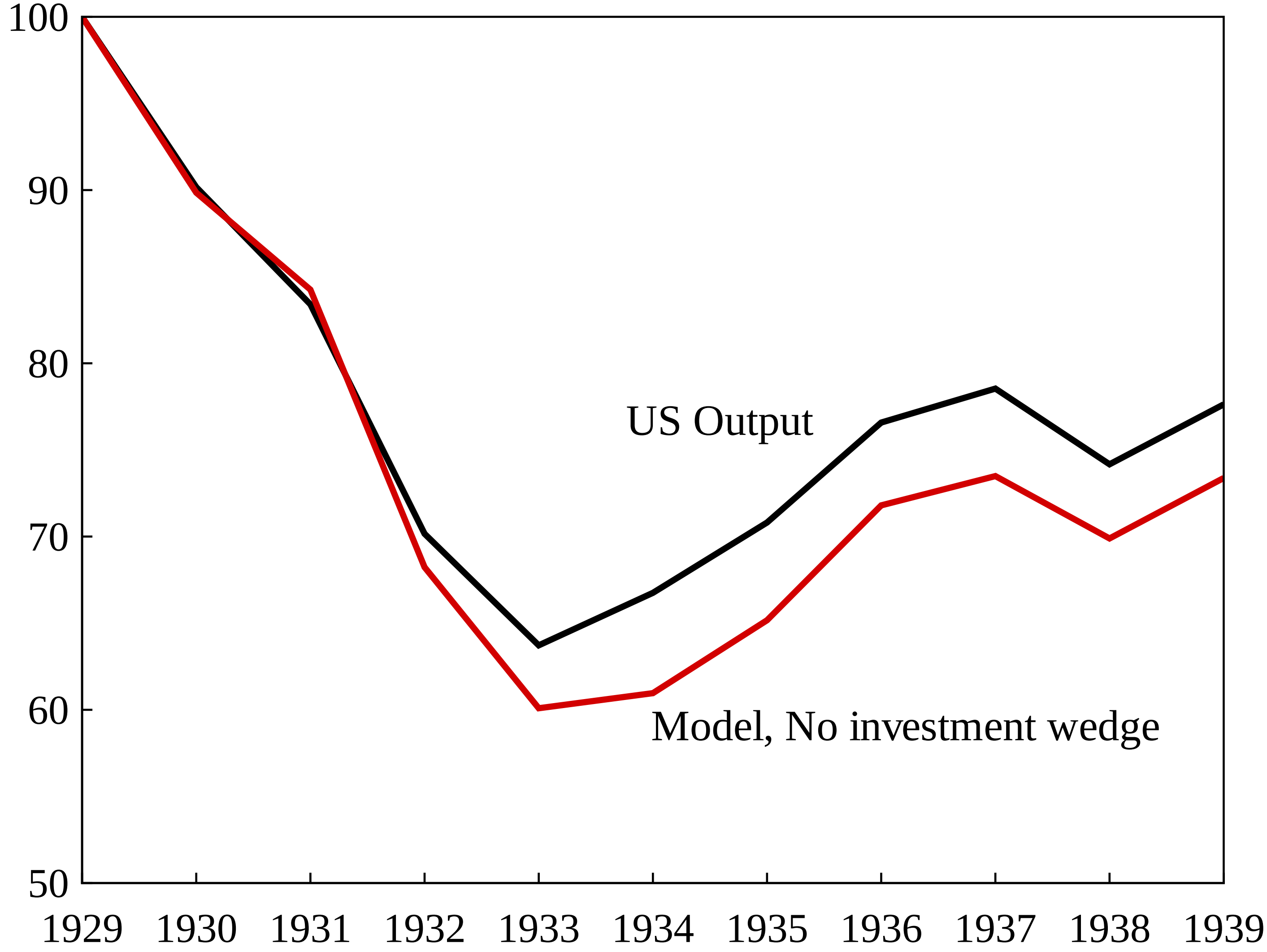
$\Rightarrow$ Implies many existing theories not promising, e.g.,

- Models with agency costs
- Models with collateral constraints

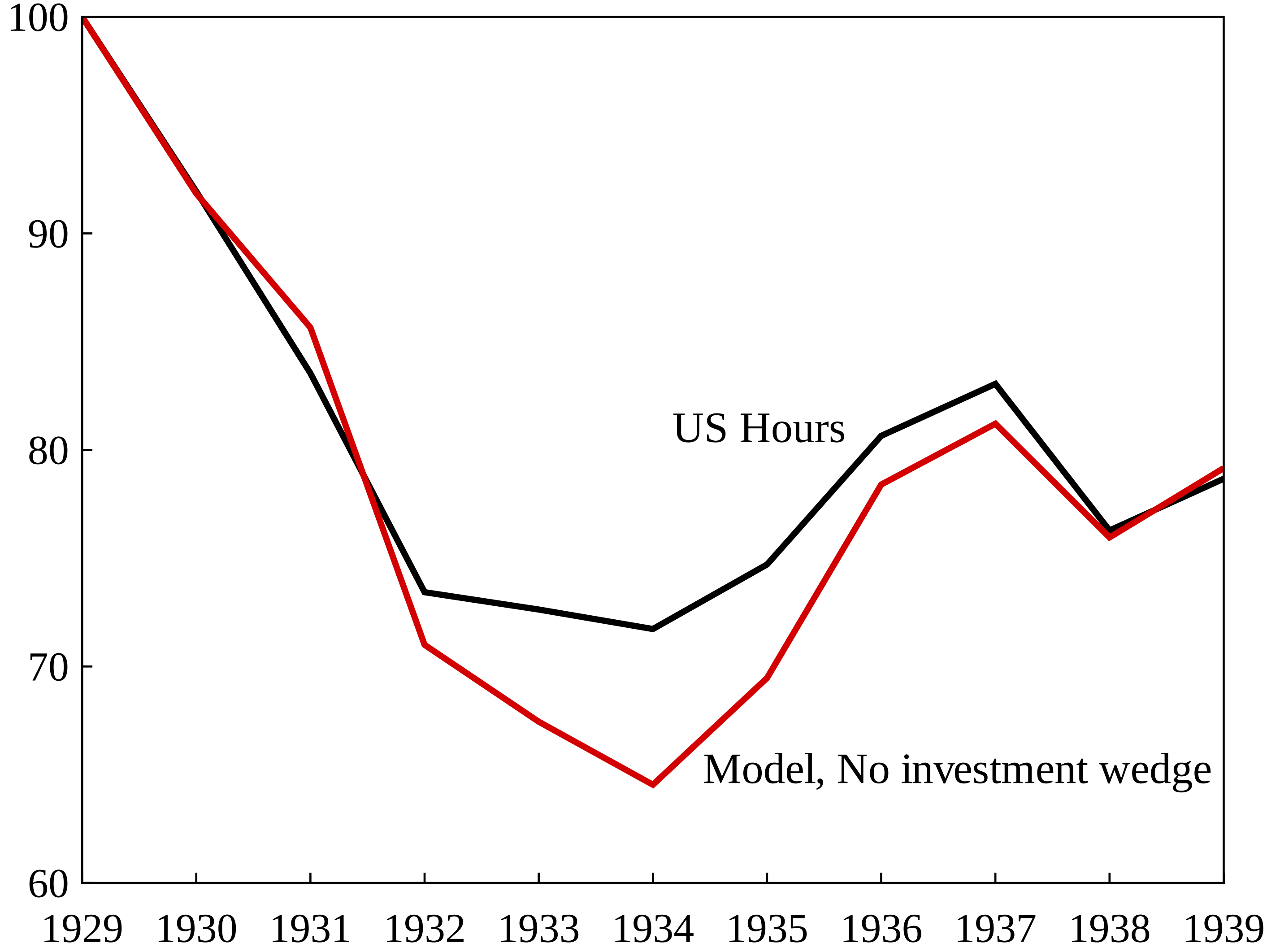
Wedges for US Great Depression



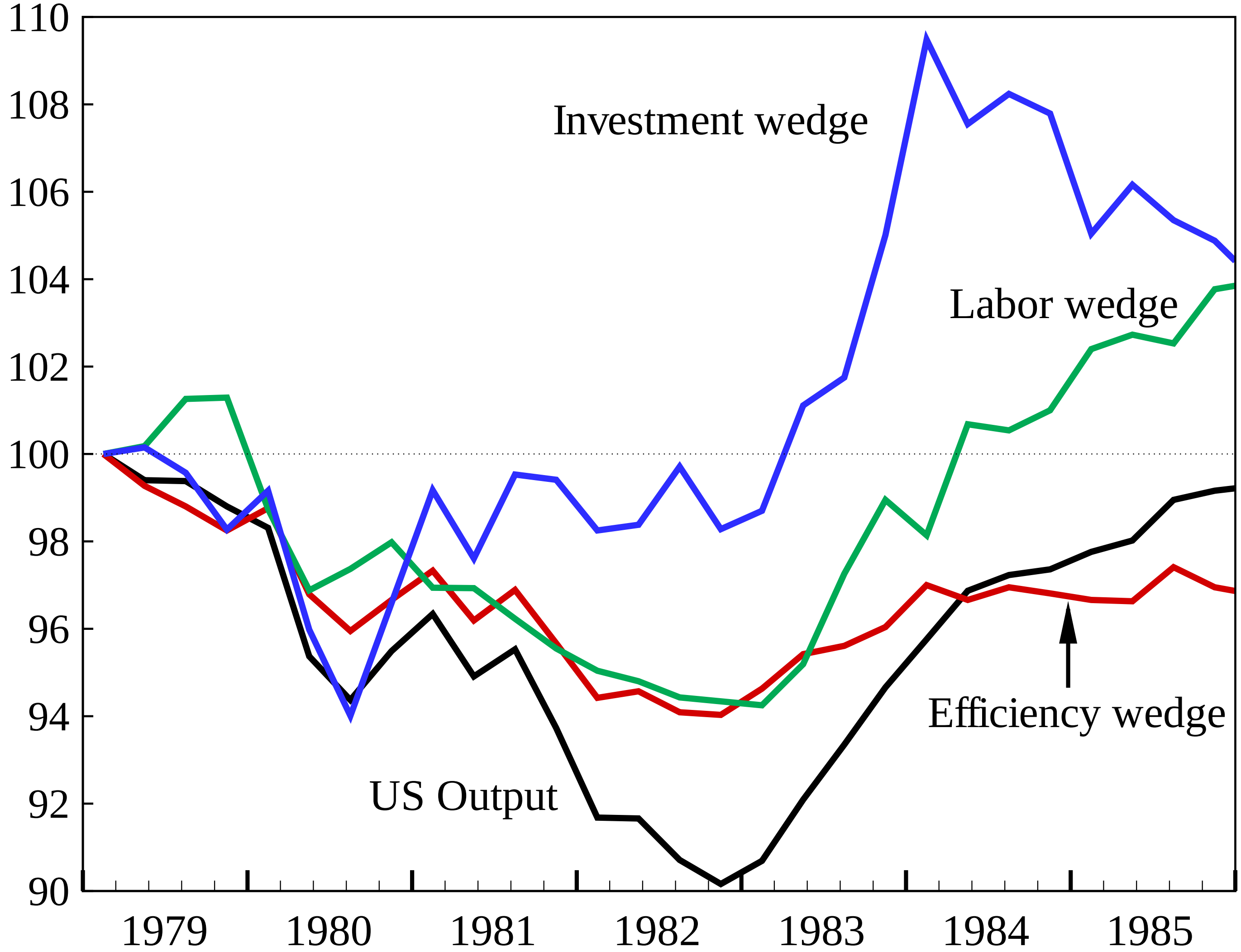
Predicted Output without Investment Wedge



Predicted Hours without Investment Wedge



Wedges for US 1980s Recession



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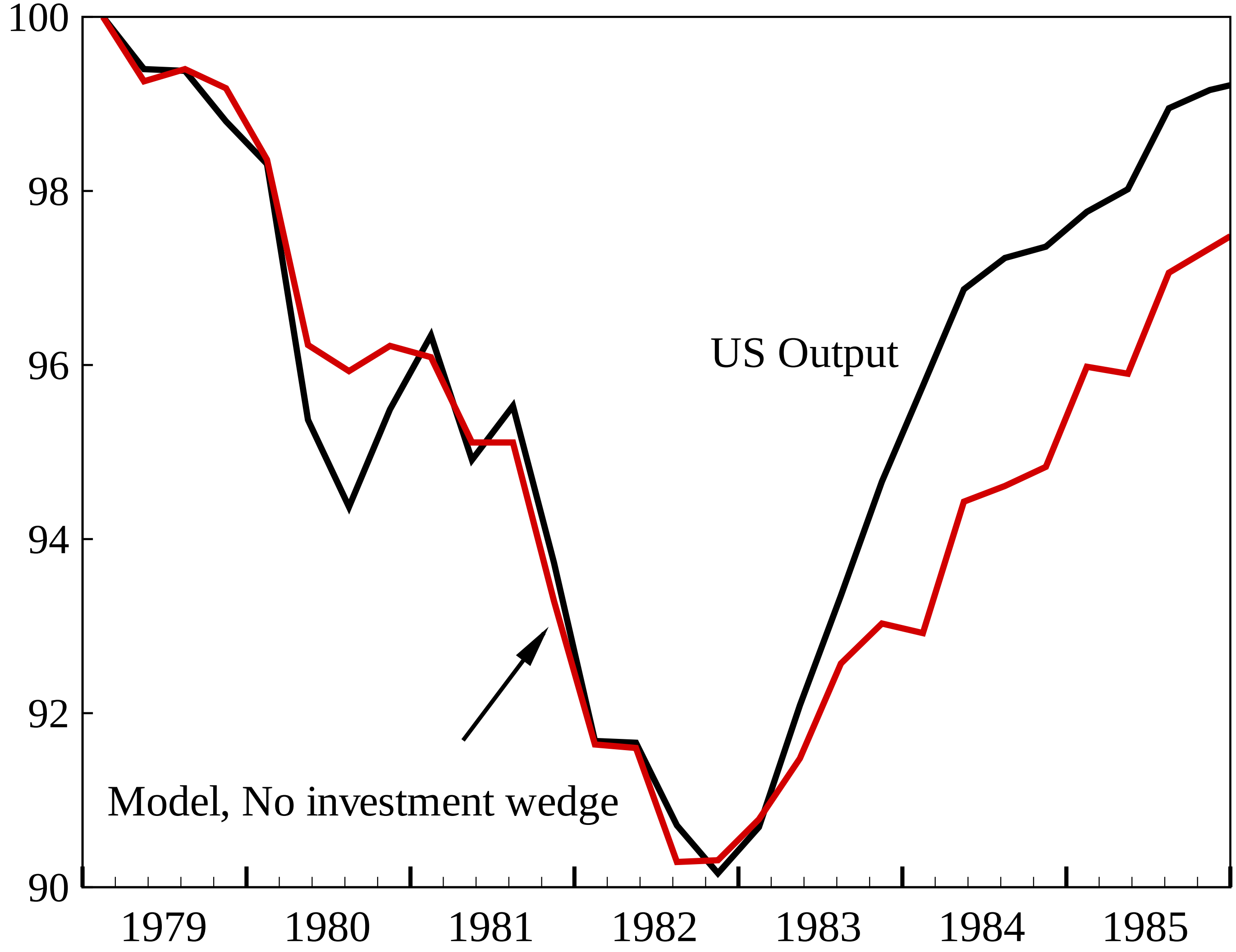

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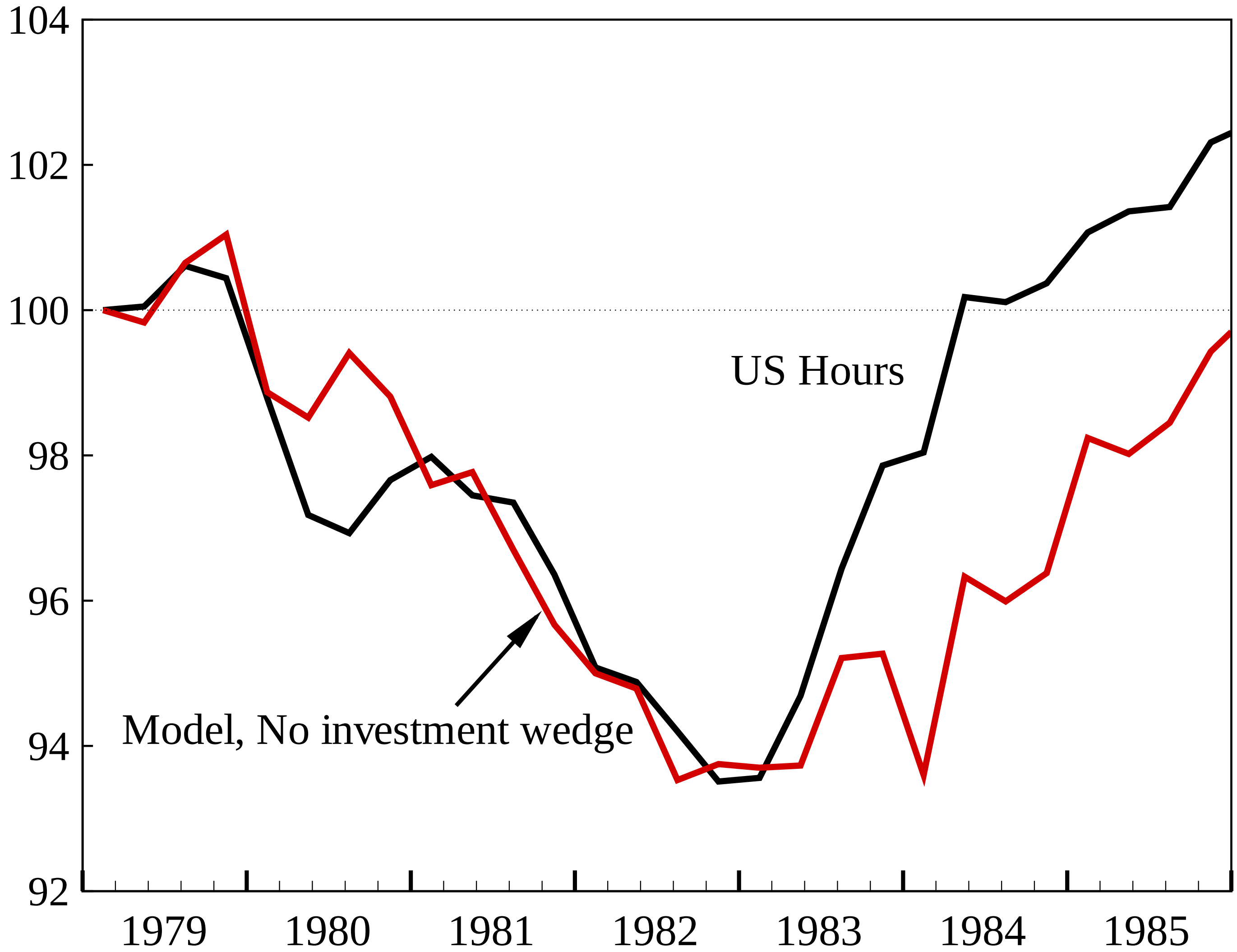

Christiano and Davis Critiques of BCA

Critique 1: Results Sensitive to Capital Tax Choice

- Original budget constraint with wedge τ_{xt}

$$c_t + (1 + \tau_{xt})x_t \leq (1 - \tau_{lt})w_t l_t + r_t k_t + T_t$$

- Alternative budget constraint with wedge τ_{kt}

$$c_t + k_{t+1} - k_t \leq (1 - \tau_{lt})w_t l_t + (1 - \tau_{kt})(r_t - \delta)k_t + T_t$$

- Christiano-Davis claim: results sensitive to choice of τ_{xt}, τ_{kt}

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  Are they right?

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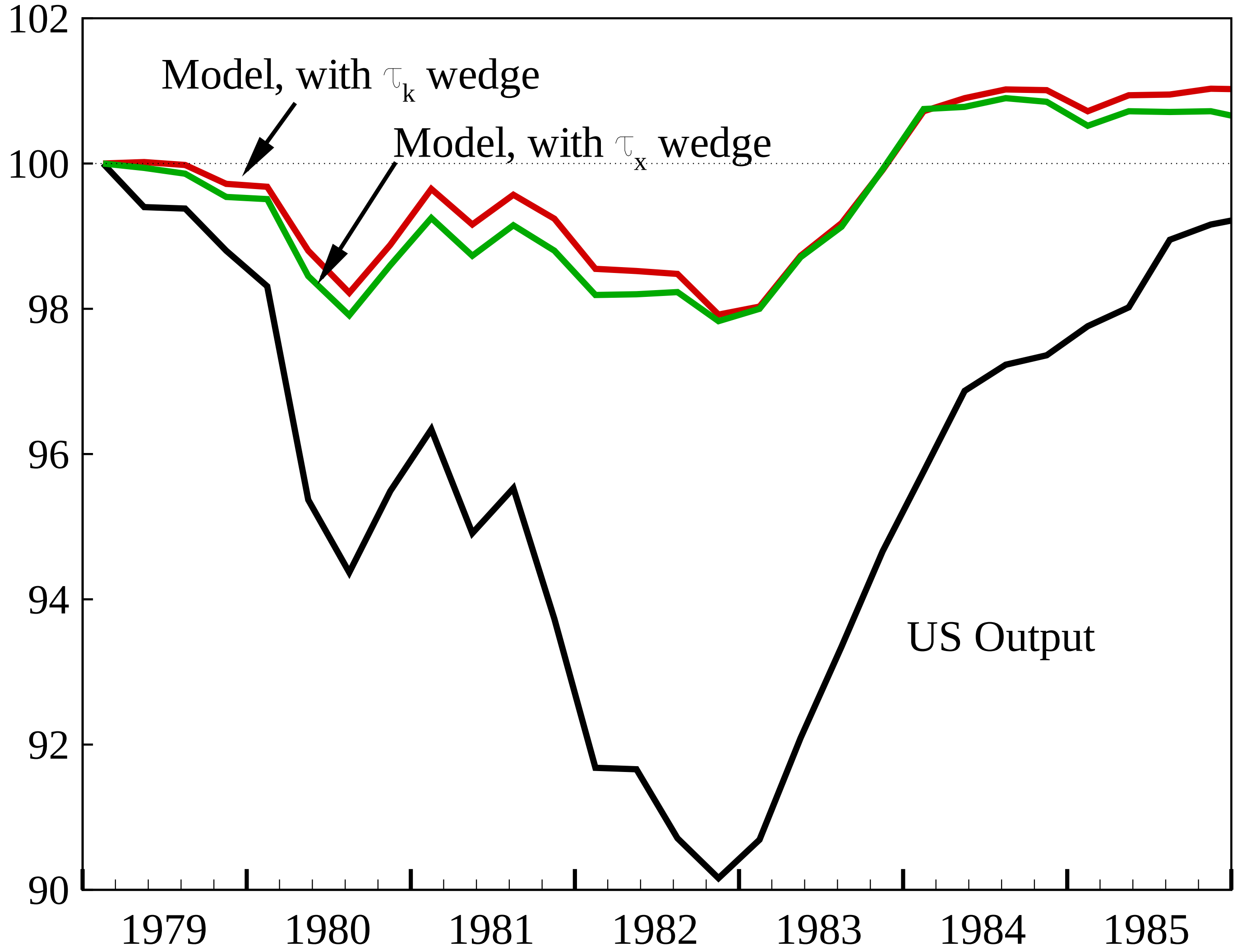
- Christiano-Davis claim: results sensitive to choice of τ_{xt}, τ_{kt}

Are they right? *No!*

CKM Response (Fed Staff Report 384)

- Theoretically, the two economies are equivalent
- Numerically,
  - Can differ slightly if FOCs linearized
  - But find tiny difference even with extreme adjustment costs

Nearly-Equivalent Model Predictions



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- Why did CD find a difference?

CKM Response (Fed Staff Report 384)

- Theoretically, the two economies are equivalent
- Numerically,
  - Can differ slightly if FOCs linearized
  - But find tiny difference even with extreme adjustment costs
- Why did CD find a difference?
  - Answer: they didn't fix expectations

Need to Keep Expectations Fixed

- Let $s_t = [s_{1t}, s_{2t}, s_{3t}, s_{4t}]$ be *latent* state vector

$$s_{t+1} = P_0 + P s_t + Q \epsilon_{t+1}$$

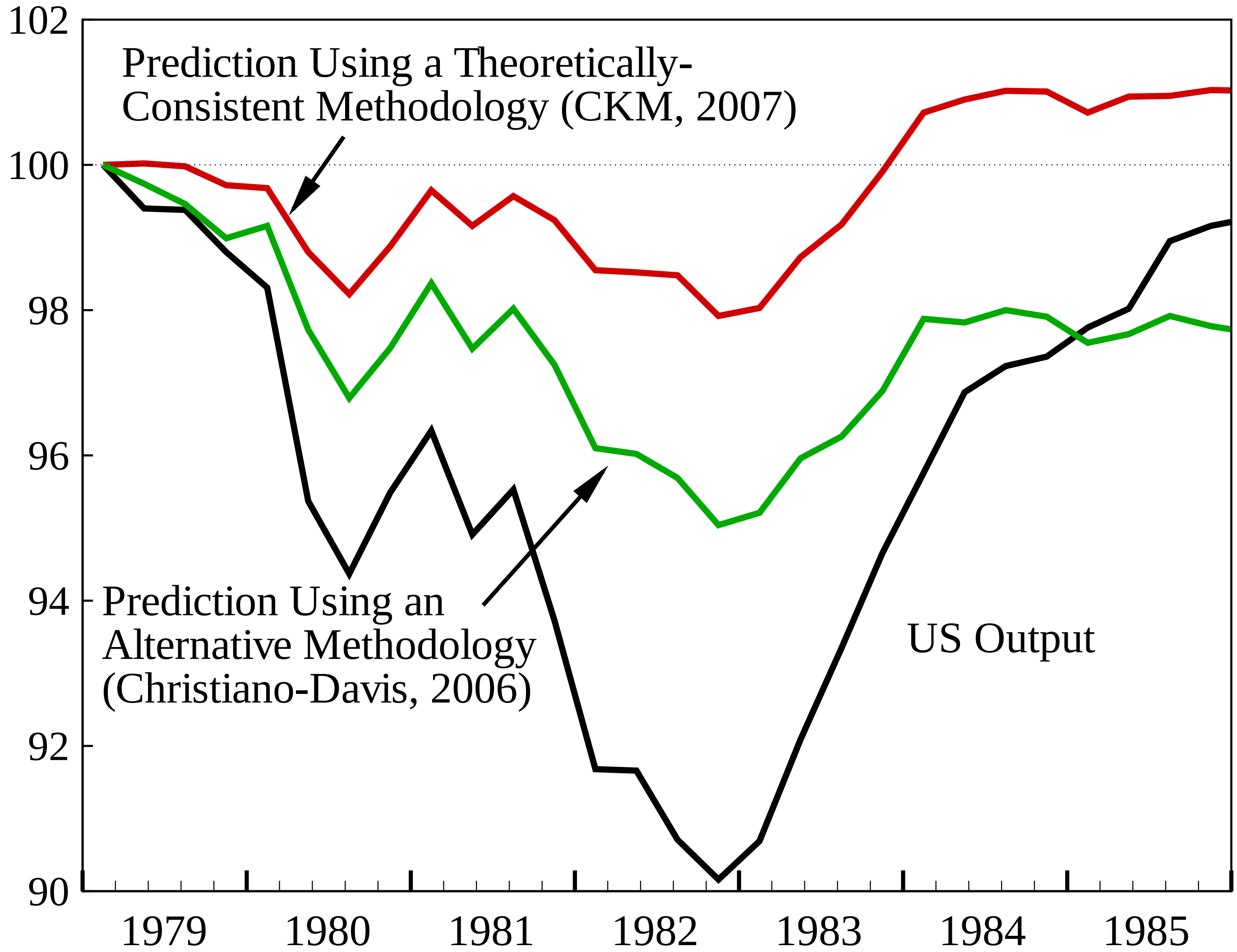
- In practice, associate wedges with elements of s_t:

$$\log A(s^t) = s_{1t},\ \tau_l(s^t) = s_{2t},\ \tau_x(s^t) = s_{3t},\ \log g(s^t) = s_{4t}$$

- For one-wedge contribution, say, of efficiency wedge:

$$\log A(s^t) = s_{1t},\ \tau_l(s^t) = \bar{\tau}_l,\ \tau_x(s^t) = \bar{\tau}_x,\ \log g(s^t) = \log \bar{g}$$

Need Theoretically-Consistent Expectations



Critique 2: BCA Ignores "Spillover Effects"

- CD use VAR approach
  - Find financial friction shock important for business cycles
  - Argue the finding is inconsistent with BCA results
- CKM use BCA approach
  - Find investment wedge plays small role for business cycles
  - Argue that CD finding is consistent with BCA results

Critique 2: BCA Ignores "Spillover Effects"

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- How can it be consistent?

Critique 2: BCA Ignores "Spillover Effects"

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  - Find investment wedge plays small role for business cycles
  - Argue that CD finding is <u>consistent</u> with BCA results
- How can it be consistent?
  - VAR sums effects of a particular shock acting on all wedges
  - BCA sums movements in investment wedge due to all shocks

Popular Alternative to BCA: Structural VARs

Current Practice: Structural VARs

- Provide summaries of facts to guide theorists, e.g.,
  - What happens after a technology shock?
  - What happens after a monetary shock?

- Impulse responses used to identify promising classes of models, e.g.,
  - If SVAR finds positive technology shock leads to fall in hours
  - Points to sticky price models (not RBC models) as promising

- SVARs are used a lot . . . but are they useful guides for theory?

An Evaluation of SVARs Using Growth Model

- Use prototype growth model
- Plot theoretical impulse response from model
- Generate data from model and apply SVAR procedure
- Plot empirical impulse response identified by SVAR procedure
- Compare responses

Main Findings for SVAR Study

- Using growth model with SVAR assumptions met
- Asking, What happens after technology shock?
- Find:
  - SVAR procedure does not uncover model's impulse response
  - Having capital in model requires infeasibly many VAR lags
  - Earlier equivalence results imply that SVARs are not useful guides

What You Get from SVAR Procedure

- Structural MA

$$X_t = A_0\epsilon_t + A_1\epsilon_{t-1} + A_2\epsilon_{t-2} + \ldots,\ E\epsilon_t\epsilon_t' = \Sigma$$

What You Get from SVAR Procedure

- Structural MA for ϵ=['technology shock', 'demand shock']$'$

$$X_t = A_0\epsilon_t + A_1\epsilon_{t-1} + A_2\epsilon_{t-2} + \ldots, \ E\epsilon_t\epsilon_t' = \Sigma$$

where $X_t = [\Delta$ Log labor productivity, $(1-\alpha L)$Log hours$]'$

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where X_t = [Δ Log labor productivity, $(1 - \alpha L)$Log hours]$'$

- Identifying assumptions:
  - Technology and demand shocks uncorrelated ($\Sigma = I$)
  - Demand shock has no long-run effect on productivity

- Impulse response from structural MA:

  Blip ϵ_1^d for response of productivity to demand

$$\log(y_1/l_1) - \log(y_0/l_0) = A_0(1,2)$$

$$\log(y_2/l_2) - \log(y_0/l_0) = A_0(1,2) + A_1(1,2)$$

$$\vdots$$

$$\log(y_t/l_t) - \log(y_0/l_0) = A_0(1,2) + A_1(1,2) + \ldots + A_t(1,2)$$

- Long-run restriction:

  Demand shock has no long run effect on level of productivity

$$\sum_{j=0}^{\infty} A_j(1,2) = 0$$

Deriving Structural MA from VAR

- OLS regressions on bivariate VAR: $B(L)X_t = v_t$

$$X_t = B_1 X_{t-1} + B_2 X_{t-2} + B_3 X_{t-3} + B_4 X_{t-4} + v_t, \quad E v_t v_t' = \Omega$$

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$$X_t = B_1 X_{t-1} + B_2 X_{t-2} + B_3 X_{t-3} + B_4 X_{t-4} + v_t, \quad E v_t v_t' = \Omega$$

- Invert to get MA: $X_t = B(L)^{-1} v_t = C(L) v_t$

$$X_t = v_t + C_1 v_{t-1} + C_2 v_{t-2} + \ldots$$

with $C_j = B_1 C_{j-1} + B_2 C_{j-2} + \ldots + B_j, \quad j = 1, 2, \ldots$

Identifying Assumptions

- Work from $X_t = v_t + C_1 v_{t-1} + C_2 v_{t-2} + \ldots,\ E v_t v_t' = \Omega$

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with $A_0 \epsilon_t = v_t$, $A_j = C_j A_0$, $A_0 \Sigma A_0' = \Omega$

Identifying Assumptions

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with $A_0 \epsilon_t = v_t,\ A_j = C_j A_0,\ A_0 \Sigma A_0' = \Omega$

- Identifying assumptions determine 7 parameters in A_0, Σ
  - Structural shocks ϵ are orthogonal, $\Sigma = I$
  - Demand shocks have no long-run effect on labor productivity

Identifying Assumptions

- Work from $X_t = v_t + C_1 v_{t-1} + C_2 v_{t-2} + \ldots$, $E v_t v_t' = \Omega$
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- Identifying assumptions determine 7 parameters in A_0, Σ
  - Structural shocks ϵ are orthogonal, $\Sigma = I$
  - Demand shocks have no long-run effect on labor productivity

$\Rightarrow$ 7 equations ($A_0 \Sigma A_0' = \Omega$, $\Sigma = I$, $\sum_j A_j(1,2) = 0$)

Use Growth Model Satisfying 3 Key SVAR Assumptions

- Only 2 shocks ("technology," "demand")
- Shocks are orthogonal
- Technology shock has unit root, demand shock does not

This is the best case scenario for SVARs

Specification of Shocks in the Model

- Technology shock is efficiency wedge $A = z^{1-\theta}$

$$\log z_t = \mu_z + \log z_{t-1} + \eta_{zt}$$

- "Demand" shock is labor wedge

$$\tau_{lt} = (1-\rho)\bar{\tau}_l + \rho\tau_{lt-1} + \eta_{\tau t}$$

- With 3 key SVAR assumptions imposed
  - Only 2 shocks ("technology," "demand")
  - Shocks orthogonal ($\eta_z \perp \eta_\tau$)
  - Technology shock has unit root, demand shock does not

Our Evaluation of SVAR Procedure

- Use growth model satisfying SVAR's 3 key assumptions
- Model has **theoretical** impulse response

$$X_t = D(L)\eta_t$$

- Generate many sequences of data from model
- Apply SVAR to these data to get **empirical** impulse response

$$X_t = A(L)\epsilon_t$$

- Compare model impulse responses with SVAR responses

Three Possible Problems

1. Noninvertibility when $\alpha = 1$

2. Small samples (around 250 quarters)

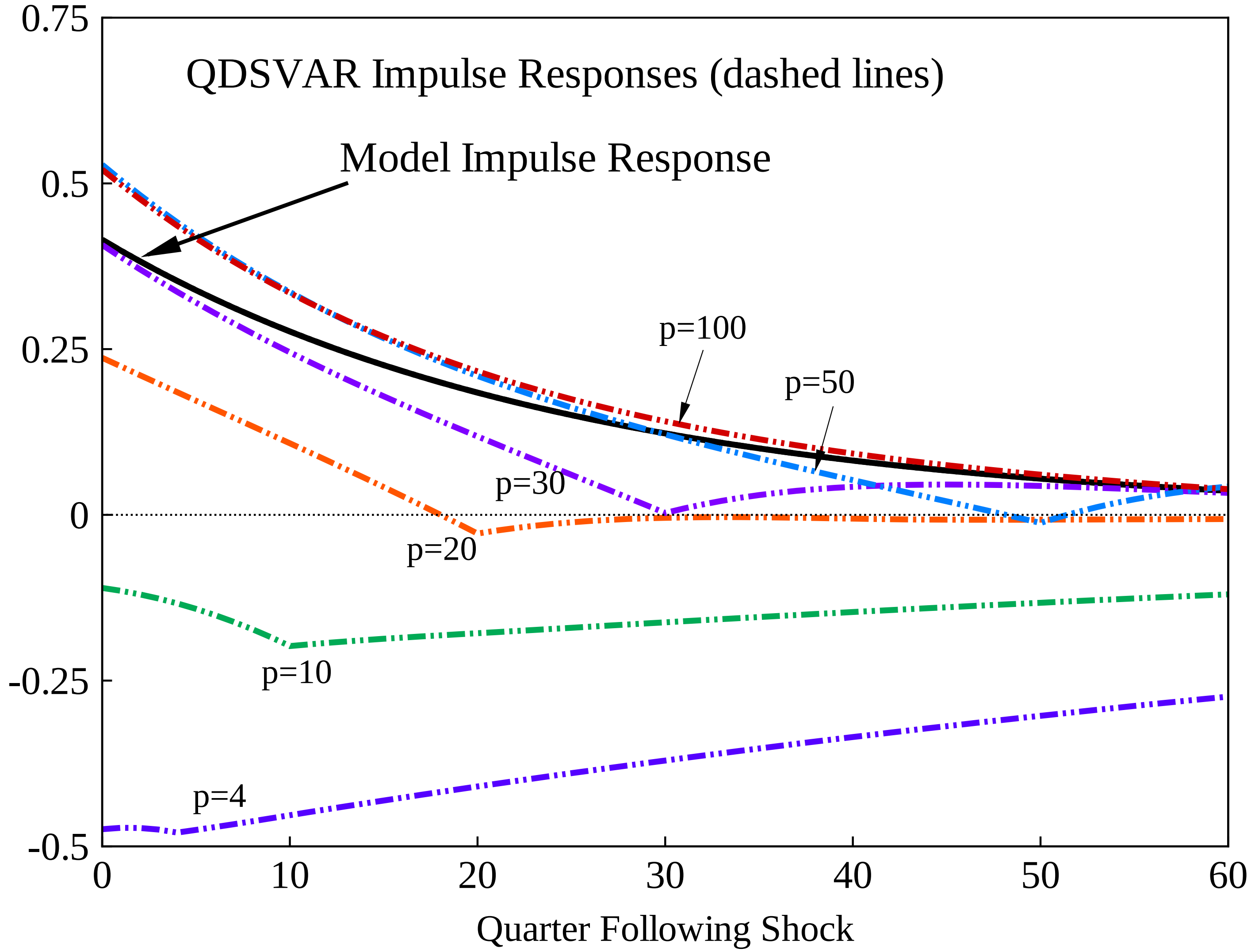
3. Short lag length

$\#3$ is quantitatively most important

The Short Lag Length Problem

- Using
  - Quasi-differencing (QDSVAR) to avoid invertibility problem
  - 100,000 length sample to avoid small sample problem

... still cannot uncover model's impulse response


QDSVAR Impulse Responses (dashed lines)
Model Impulse Response
p=100
p=50
p=30
p=20
p=10
p=4
0.75
0.5
0.25
0
-0.25
-0.5
0
10
20
30
40
50
60
Quarter Following Shock

Long AR Needed Because of Capital

- Capital decision rule, with $\hat{k}_t = k_t / z_{t-1}$:

$$\log \hat{k}_{t+1} = \gamma_k \log \hat{k}_t + \gamma_z \eta_{zt} + \gamma_\tau \tau_t$$

- So others, like l_t, have ARMA representation

$$\log l_t = \gamma_k \log l_{t-1} + \phi_z (1 - \kappa_z L) \eta_{zt} + \phi_\tau (1 - \kappa_\tau L) \tau_t$$

- What does the AR representation, $B(L) X_t = v_t$, look like?

Model Has Infinite-order AR

- Proposition: Model has VAR coefficients B_j such that

$$B_j = MB_{j-1}, \quad j \geq 2,$$

where M has eigenvalues equal to α (the differencing parameter) and

$$\left(\frac{\gamma_k - \gamma_l \phi_k / \phi_l - \theta}{1 - \theta}\right)$$

γ_k, γ_l are coefficients in the capital decision rule

ϕ_k, ϕ_l are coefficients in the labor decision rule

- Eigenvalues of M are α and .97 for the baseline parameters

What Happens with Two Few Lags

- From SVAR procedure, want to recover model's:
  - Variance-covariance matrix Ω_m
  - Sum of MA coefficients $\bar{C}_m$

- Example: Run VAR with 1 Lag and see what SVAR recovers
  - Variance-covariance matrix (with $V(X) = EXX'$):

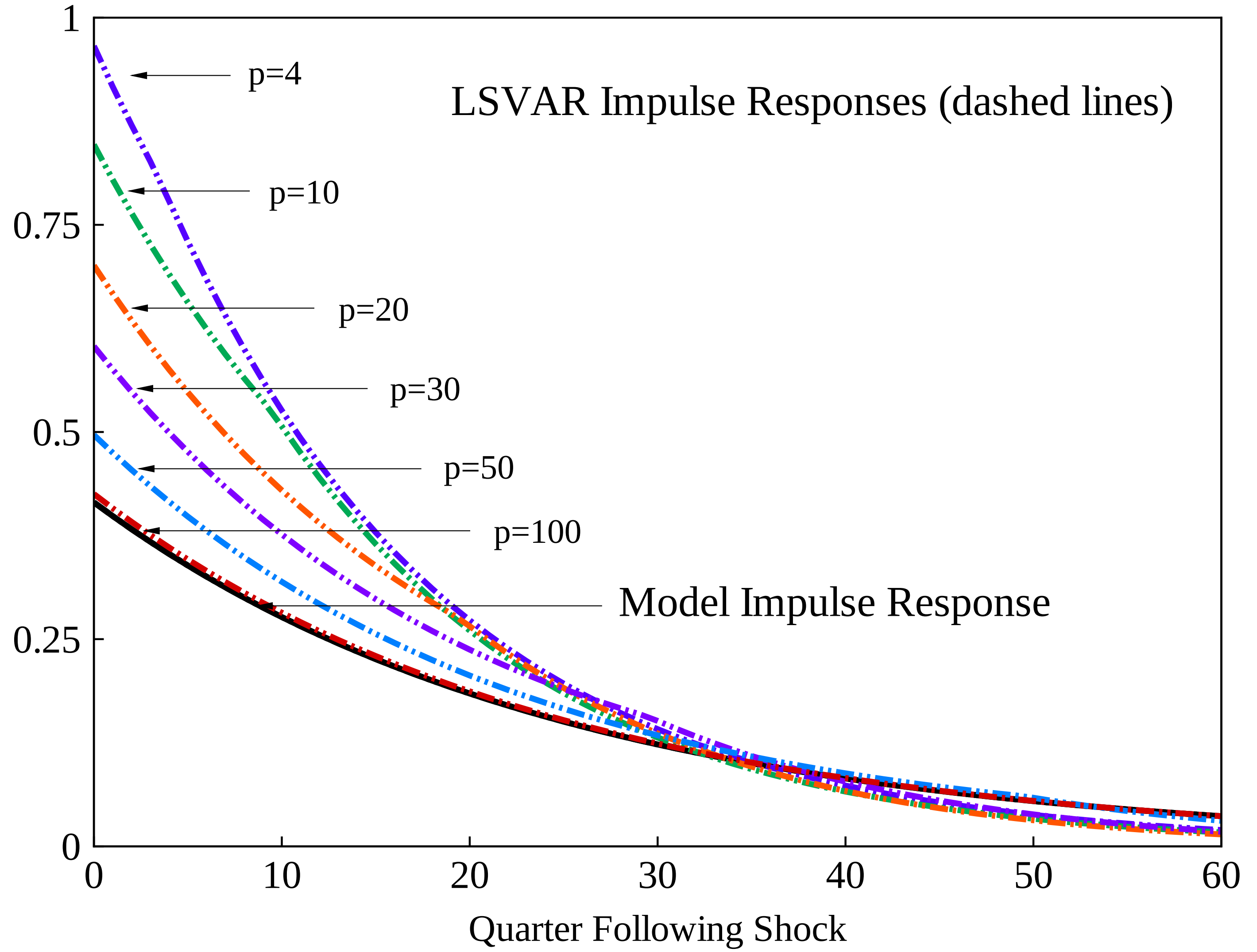
$$\Omega = \Omega_m + M\left(\Omega_m - \Omega_m V(X)^{-1}\Omega_m\right)M'$$

  - (Inverse of) sum of MA coefficients:

$$\bar{C}^{-1} = \bar{C}_m^{-1} + M(I-M)^{-1}C_{m,1} + M\left(\Omega_m - V(X)\right)V(X)^{-1}$$

Notice that M is important factor in garbled terms!

Problems Still Arise if Hours in Levels



Recap of Lecture I

- BCA is a promising alternative to SVARs

- Statistical methods must be guided by theory

- Empirical "facts" may indeed be fictions

II. Beyond BCA: Some Applications

Background

- Connecting the dots...
  - Hours boomed in 1990s while wages fell
  - Very puzzling since
    - aggregate TFP was not above trend
    - labor taxes were relatively high

    $\Rightarrow$ CKM would recover large labor wedge

- This puzzled us for years

Meanwhile...

- Working on projects related to
  - Stock market boom
  - Financial account collapse

- Key factor for both is intangible capital...

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- Working on projects related to
  - Stock market boom
  - Financial account collapse

- Key factor for both is intangible capital...

  ... which we later discovered results in a labor wedge

What is Intangible Capital?

- Accumulated know-how from investments in
  - R&D
  - Software
  - Brands
  - Organization know-how

  that are *expensed* by firms

US Stock Market Boom

Stock Market Boom

- Value of US corporations doubled between 1960s and 1990s
- We asked,
  - Was the stock market overvalued in 1999?
  - Why did the value double?

Was Market Overvalued in 1999?

- Many concluded it was based on earnings-price (E/P) ratio
- But, E/P is not the return if firm invests in intangible capital
- Needed a way to measure intangible capital

Three Ways to Measure Intangible Capital

- Residually: $V - qK_T$
- Directly with estimates of:
  - Expenditures (R&D+software+ads+org capital)
  - Depreciation rates
- Indirectly with estimates of:
  - Tangible capital stocks
  - NIPA profits = tangible rents + intangible rents − intangible expenses

- Corporate value = present value of discounted distributions

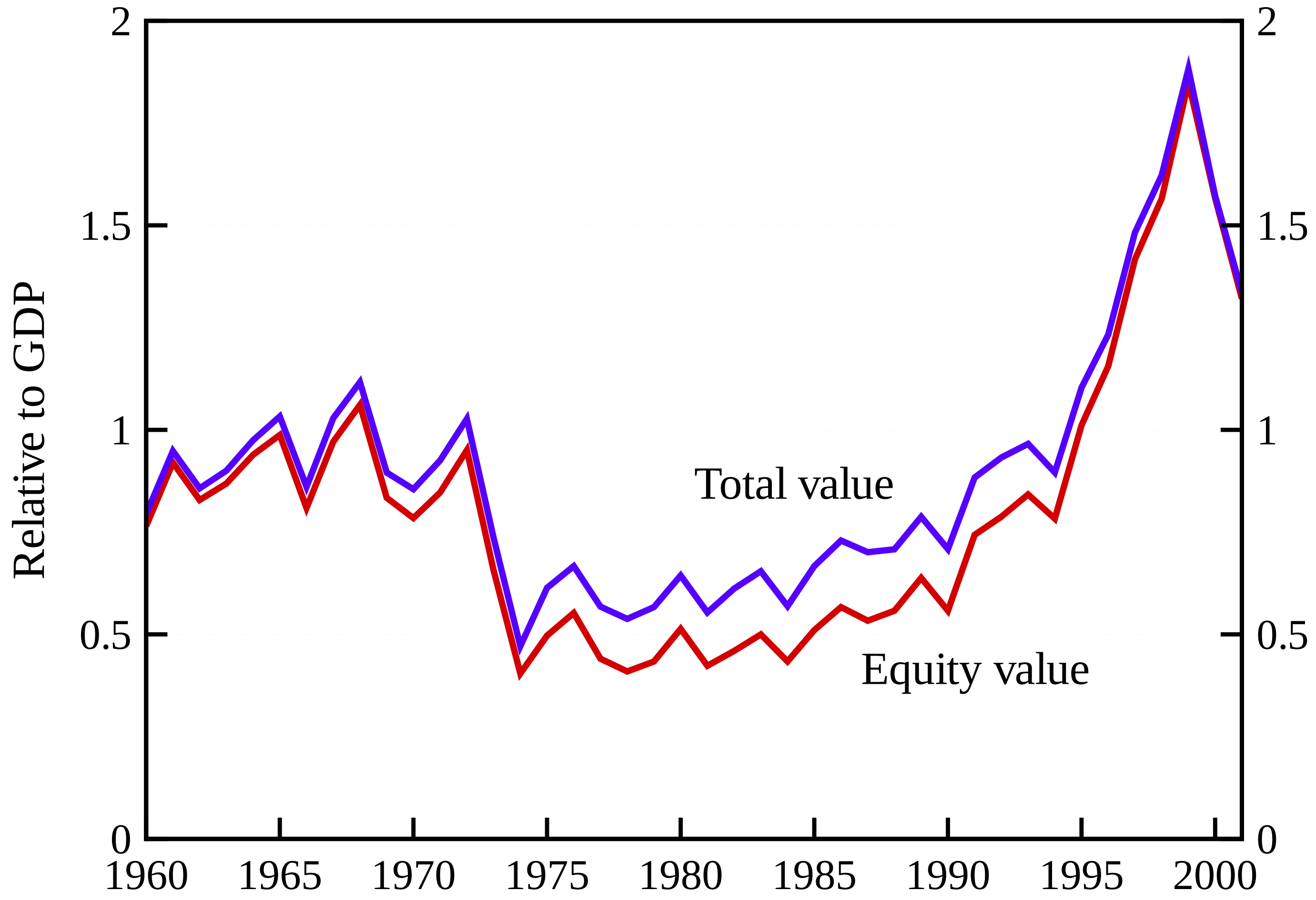
  = value of productive capital

$$V_t = \sum_i \Big\{ \underbrace{q_{T,i,t} K_{T,i,t+1}}_{\text{Tangible}} + \underbrace{\underbrace{q_{I,i,t} K_{I,i,t+1}}_{\text{Plant-specific}} \Big\} + \underbrace{q_{M,t} K_{M,t+1}}_{\text{Global}}}_{\text{Intangible}}$$

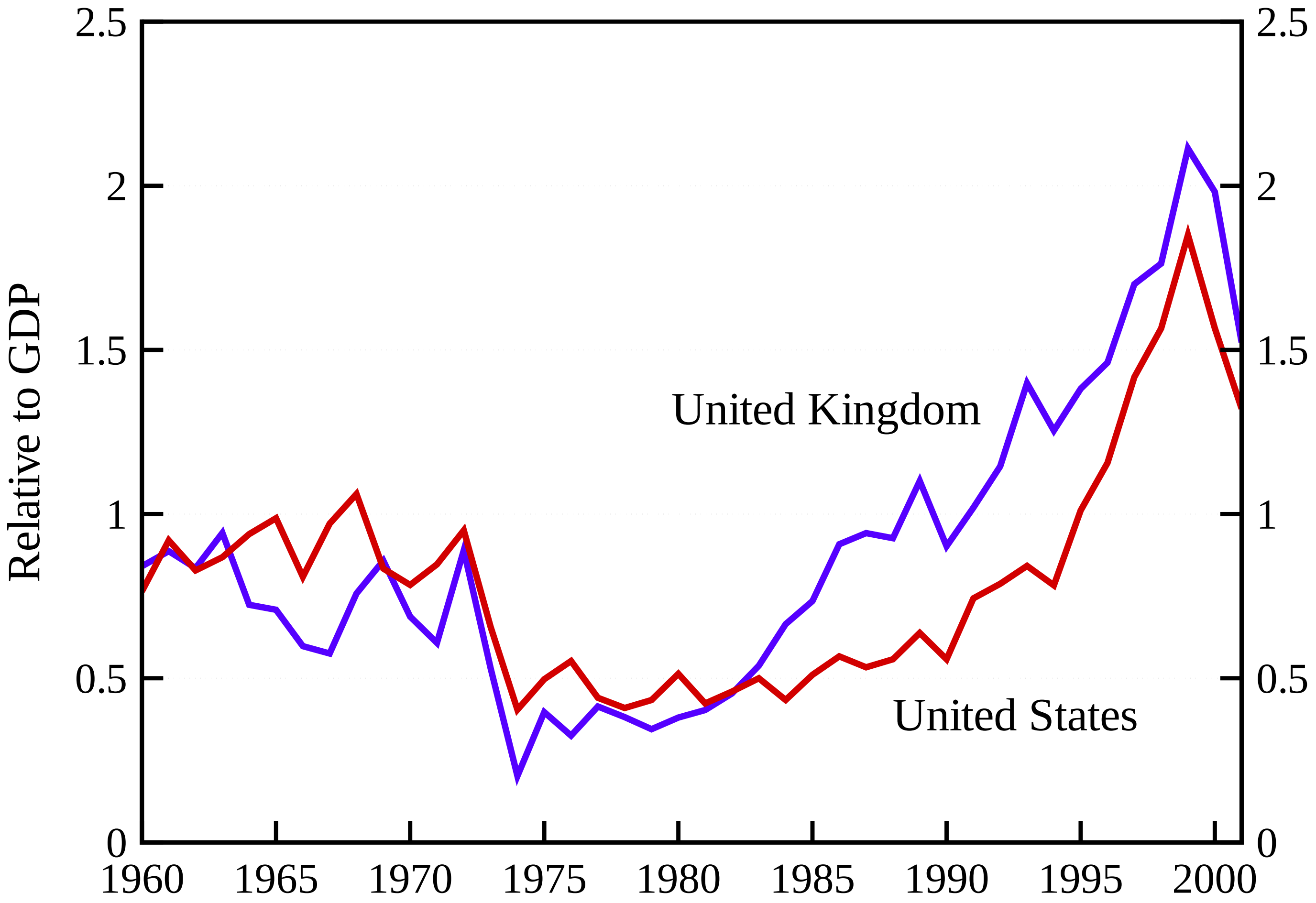
  where i indexes countries

- With only domestic tangible capital, theory fails miserably!

Dramatic Rise in US Values, But K_T/GDP≈ 1



Dramatic Rise in Both US and UK



Theory Yields Some Surprising Results

- Stock values *should* have been high in the 1990s and were.

- Values to GDP *should* have doubled between the 60s and 90s and did

- PE ratios *should* have doubled over the same period and did

What Drives the Results?

- Significant changes in prices of capital (q's)

- Preferences:

$$\sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t) N_t$$

- Technologies:

$$y_{1,t} = f^c(k_{1T,t}, k_{1I,t}, z_t n_{1,t})$$ 1=corporate, T,I=tangible,intangible

$$y_{2,t} = f^{nc}(k_{2,t}, z_t n_{2,t})$$ 2=noncorporate

$$y_t = F(y_{1,t}, y_{2,t})$$

Variables:

c = consumption, ℓ = leisure, N = household size

y = output, k = capital, n = labor, z = technology

- and the Corporation:

$$\max \sum_{t=0}^{\infty} p_t \{ p_{1,t} y_{1,t} - w_t n_{1,t} - x_{1T,t} - x_{1I,t} \\ - \tau_{1,t} \left[p_{1,t} y_{1,t} - w_t n_{1,t} - \delta_{1T} k_{1T,t} - \tau_{1k,t} k_{1T,t} - x_{1I,t} \right] \\ - \tau_{1k,t} k_{1T,t} \}$$

- and the Household (no capital gains case):

$$\sum_{t=0}^{\infty} p_t \{ (1 + \tau_{c,t}) c_t + V_{1s,t}(s_{1,t+1} - s_{1,t}) + V_{2s,t}(s_{2,t+1} - s_{2,t}) + V_{b,t} b_{t+1} \} \\ \leq \sum_{t=0}^{\infty} p_t \{ (1 - \tau_{d,t}) d_{1,t} s_{1,t} + d_{2,t} s_{2,t} + b_t + (1 - \tau_{n,t}) w_t n_t + \kappa_t \}$$

Main Theoretical Result

$$V_t = (1 - \tau_{dt})\left[k_{1T,t+1} + (1 - \tau_{1t})k_{1I,t+1}\right]$$

V	value of corporate equities
τ_d	tax rate on dividends
k_{1T}	tangible corporate capital stock
τ_1	tax rate on corporate income
k_{1I}	intangible corporate capital stock

Main Theoretical Result

$$V_t = (1 - \tau_{dt})\left[k_{1T,t+1} + (1 - \tau_{1t})k_{1I,t+1}\right]$$

- *Proposition.*
  - If τ_{dt} constant and revenues lump-sum rebated,
  - then capital-output ratios independent of τ_d

  Proof. τ_d drops out of intertemporal condition

- *Corollary.* Periods of high τ_d have low $V/$GDP and vice versa

Therefore...

- Stock values *should* have been high in the 1990s and were.
- Values to GDP *should* have doubled between the 60s and 90s and did
- Values to GDP *should* have doubled between the 60s and 90s and did
- PE ratios *should* have doubled over the same period and did

Taxes—affecting q's—and Intangibles Important

	1960-69	1998-01
Predicted fundamental values		
Domestic tangible capital	.56	.84
Domestic intangible capital	.23	.35
Foreign capital	.09	.38
Total relative to GDP	.88	1.57
Price-earnings ratio	13.5	27.5
Actual values		
Corporate equities	.90	1.58
Net corporate debt	.04	.03
Total relative to GDP	.94	1.60
Price-earnings ratio	14.5	28.1

Recap of Stock Market Study

- Value of US corporations doubled between 1960s and 1990s
- We asked,
    - Was the stock market overvalued in 1999?
    - Why did the value double?

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- We asked,

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  - Why did the value double?

    — effective taxes on corporate distributions fell

- And, we found that intangible capital is important factor

Went From One Puzzle to the Next...

Our Estimate of Foreign Capital Value

- Since US multinationals do significant FDI,
  - Computed estimate of value
  - After the fact, we compared them

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Our Estimate of Foreign Capital Value

- Since US multinationals do significant FDI,
  - Computed estimate of value—not realizing BEA provides one
  - After the fact, we compared them

Were the BEA and our estimates close? *No!*

Stumbled Upon Another Puzzle

- For US subsidiaries, BEA reports
  - Small value for capital abroad
  - Large value for profits from abroad

$\Rightarrow$ Large return to DI of US

- For foreign subsidiaries in US, BEA reports
  - Small value for capital in US
  - Really small value for profits

$\Rightarrow$ Small return to DI in US

The Return Differential

- BEA reports for 1982–2006:
  - US companies earned 9.4% average returns
  - Foreign companies earned 3.2% average returns

on their foreign direct investment abroad

What Could Account for Return Differential?

- Multinational have large intangible capital stocks
  - DI profits include intangible rents (+) less expenses (−)
  - DI stocks don't include intangible capital

  ⇒ BEA returns not equal economic returns

- FDI in US is negligible until late 1970s

  ⇒ Timing of investments different in US & ROW

To Interpret the Data

- Need to consider nature of intangibles
  - Rival versus nonrival
  - Expensed at home versus abroad

- Want theory that incorporates these

Extensions to Neoclassical Theory

- Add two types of intangible capital

  1. Rival that is plant-specific (K_I)
  2. Nonrival that is firm-specific (M)

- Add locations since technology capital nonrival (N)

- To otherwise standard multi-country DSGE model

A Useful Example

- US drug company with employees
  - Bob who develops a new drug in NC
  - 50 drug reps at 50 US locations
  - 2 drug reps at 2 Belgian locations
- Measuring impact of intangibles, need to keep in mind
  - Some capital is nonrival, some rival
  - Production opportunities vary with country size
  - Profits depend on timing of investments and rents

Output of Multinationals from Country j in i

$$Y_i^j = A_i \underbrace{(K_{T,i}^j)^{\alpha_T}(L_i^j)^{1-\alpha_T}}_{\text{Tangibles}}$$

A_i : country i's TFP

Output of Multinationals from Country j in i

$$Y_i^j = A_i \underbrace{(K_{T,i}^j)^{\alpha_T}(L_i^j)^{1-\alpha_T-\alpha_I}}_{\text{Tangibles}}\underbrace{(K_{I,i}^j)^{\alpha_I}}_{\text{Add } K_I}$$

A_i : country i's TFP

Output of Multinationals from Country j in i

$$Y_i^j = A_i \underbrace{((K_{T,i}^j)^{\alpha_T} (L_i^j)^{1-\alpha_T}}_{\text{Tangibles}} \underbrace{{}^{-\alpha_I} (K_{I,i}^j)^{\alpha_I}}_{\text{Add } K_I} \underbrace{)^{1-\phi} \quad (N_i M^j)^{\phi}}_{\text{Add } M}$$

A_i : country i's TFP

N_i : country i's measure of production locations

Output of Multinationals from Country j in i

$$Y_i^j = A_i \, \underbrace{((K_{T,i}^j)^{\alpha_T} (L_i^j)^{1-\alpha_T-\alpha_I}}_{\text{Tangibles}} \underbrace{(K_{I,i}^j)^{\alpha_I}}_{\text{Add } K_I})^{1-\phi} \underbrace{(N_i M^j)^{\phi}}_{\text{Add } M}$$

(drug reps) (Bob)

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A_i : country i's TFP

N_i : country i's measure of production locations

σ_i : country i's degree of openness to FDI

Output of Multinationals from Country j in i

$$Y_i^j = A_i \underbrace{((K_{T,i}^j)^{\alpha_T}(L_i^j)^{1-\alpha_T-\alpha_I}}_{\text{Tangibles}}\underbrace{(K_{I,i}^j)^{\alpha_I}}_{\text{Add } K_I})^{1-\phi}\underbrace{\sigma_i(N_i M^j)^{\phi}}_{\text{Add } M}$$

$$\hat{A}_i^j = Y_i^j/((K_{T,i}^j)^{\alpha_T}(L_i^j)^{1-\alpha_T})$$

$$= A_i\sigma_i(K_{I,i}^j/L_i^j)^{\alpha_I(1-\phi)}(N_i M^j)^{\phi}$$

A_i : country i's TFP

N_i : country i's measure of production locations

σ_i : country i's degree of openness to FDI

$\hat{A}_i^j$: multinational j's *measured TFP* in i

Output of Multinationals from Country j in i

$$Y_i^j = A_i \underbrace{((K_{T,i}^j)^{\alpha_T} (L_i^j)^{1-\alpha_T-\alpha_I} (K_{I,i}^j)^{\alpha_I}}_{\equiv Z_i^j})^{1-\phi} \underbrace{\sigma_i (N_i M^j)^\phi}_{\text{Add } M}$$

$$\hat{A}_i^j = Y_i^j / ((K_{T,i}^j)^{\alpha_T} (L_i^j)^{1-\alpha_T})$$

$$= A_i \sigma_i (K_{I,i}^j / L_i^j)^{\alpha_I (1-\phi)} (N_i M^j)^\phi$$

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$$\hat{A}_i^j = Y_i^j / ((K_{T,i}^j)^{\alpha_T}(L_i^j)^{1-\alpha_T})$$

$$= A_i \sigma_i (K_{I,i}^j / L_i^j)^{\alpha_I(1-\phi)}(N_i M^j)^{\phi}$$

Next, aggregate over output of all multinationals j

New Aggregate Production Function

$$Y_{it} = A_{it} N_{it}^{\phi} (M_t^i + \sigma_{it}^{\frac{1}{\phi}} \sum_{j \neq i} M_t^j)^{\phi} Z_{it}^{1-\phi}$$

- Key results:
  - Output per effective person increasing in size
  - Greater openness (σ_{it}) yields *intangible* gains

Note: Size $\equiv A_i^{\frac{1}{1-(\alpha_T+\alpha_I)(1-\phi)}} N_i$

Use theory to Construct BEA Return on FDI

- Think of d=Dell, f=France

$$r_{\text{FDI},t} = (1-\tau_{p,ft})\left(Y^d_{ft} - W_{ft}L^d_{ft} - \delta_T K^d_{T,ft} - X^d_{I,ft}\right)/K^d_{T,ft}$$

$$= r_t + \underbrace{(1-\tau_{p,ft})\left[\phi + (1-\phi)\alpha_I\right]\frac{Y^d_{ft}}{K^d_{T,ft}}}_{\text{intangible rents}} - \underbrace{(1-\tau_{p,ft})\frac{X^d_{I,ft}}{K^d_{T,ft}}}_{\text{expenses}}$$

where r_t is actual return on all types of capital

What We Find

- Use model where each investment earns 4.6% on average

- We find average BEA returns on DI, 1982–2006:
  - of US $=$ 7.1% .... BEA reports 9.4%
  - in US $=$ 3.1% .... BEA reports 3.2%

$\Rightarrow$ Mismeasurement accounts for over 60% of return gap

Recap of Two Puzzles

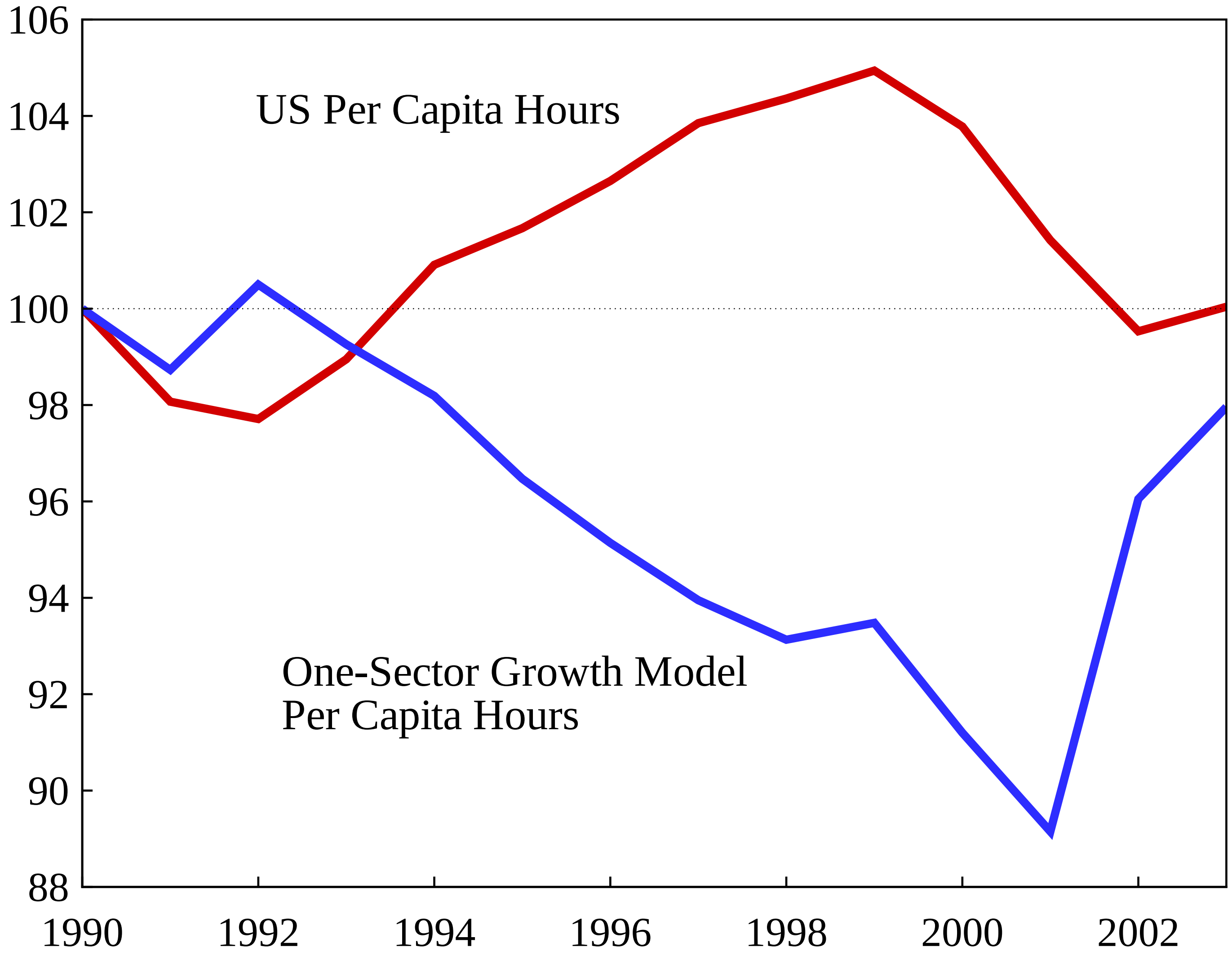
- In studying stock market boom, needed estimate of foreign capital
- Our estimates turned out to be much larger than BEA's
  - BEA returns are not equal to economic returns
  - Timing of investments different in US and ROW

Recap of Two Puzzles

- In studying stock market boom, needed estimate of foreign capital
- Our estimates turned out to be much larger than BEA's
    - BEA returns are not equal to economic returns
    - Timing of investments different in US and ROW
- Working on these projects gave us an idea for the 1990s boom

Intangible Capital and the Puzzling 1990s Boom

The Puzzle



106
104
102
100
98
96
94
92
90
88
1990
1992
1994
1996
1998
2000
2002
US Per Capita Hours
One-Sector Growth Model
Per Capita Hours

Connecting the dots...

- Previous work points to issue of mismeasurement
  - 1990s was a tech boom
  - Yet, TFP was not growing fast
  - Why? because of large intangible investments in
    - — Sweat equity
    - — Corporate R&D

Modeling the Tech Boom

- Two key factors:
  - Intangible capital that is expensed
  - Nonneutral technology change w.r.t. its production
- Idea: model tech boom as boom in intangible production

Modeling the Tech Boom

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Increased intangible investment

Modeling the Tech Boom

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  - Intangible capital that is expensed
  - Nonneutral technology change w.r.t. its production

- Idea: model tech boom as boom in intangible production

$\Rightarrow$ Increased hours in intangible production

Increased intangible investment

Understated growth in *measured* productivity

Intuition

- True productivity

$$\frac{y_t + q_t x_{It}}{h_{yt} + h_{xt}} \neq \frac{y_t}{h_{yt} + h_{xt}}$$

$$= \text{Measured productivity}$$

where

$$\begin{aligned} y_t &= \text{output of final goods and services} \\ q_t x_{It} &= \text{output of intangible production} \\ h_{yt} &= \text{hours in production of final G\&S} \\ h_{xt} &= \text{hours in production of new intangibles} \end{aligned}$$

BEA National Accounts (before 2013)

NIPA INCOME	NIPA PRODUCT
Capital consumption	Personal consumption
Taxes on production	Government consumption
Compensation **less sweat**	Government investment
Profits **less expensed**	Private tangible investment
Net interest	Net exports

Revised National Accounts

TOTAL INCOME	TOTAL PRODUCT
Capital consumption	Personal consumption
Taxes on production	Government consumption
Compensation **less sweat**	Government investment
Profits **less expensed**	Private tangible investment
Net interest	Net exports
Capital gains	**Intangible investment**

TOTAL INCOME	TOTAL PRODUCT
Capital consumption	Personal consumption
Taxes on production	Government consumption
Compensation	Government investment
Profits	Private tangible investment
Net interest	Net exports
	Intangible investment

Evidence of the Mechanism

- Macro
  - Hours boomed, but compensation per hour fell
  - GDP rose, but corporate profits fell
  - Capital gains high at end of 1990s
- Micro
  - Industry R&D boomed
  - IPO gross proceeds boomed
  - Average hours boomed selectively

Average Hours Boomed Selectively

Hours Per Noninstitutional Population Aged 16-64

	Total (1992=100)	The Educated in Select Occupations†
1992	100.0	10.3
2000	106.5	13.3
% Chg.	6.5	30.0

† Managerial, computational, and financial occupations

- Household/Business owners solve

$$\max E \sum_{t=0}^{\infty} \beta^t [\log c_t + \psi \log(1-h_t)] N_t$$

subject to

$$\begin{aligned} c_t + x_{Tt} + q_t x_{It} &= r_{Tt} k_{Tt} + r_{It} k_{It} + w_t h_t \\ &\quad - \text{taxes}_t + \text{transfers}_t + \text{nonbusiness}_t \\ k_{T,t+1} &= (1-\delta_T) k_{Tt} + x_{Tt} \\ k_{I,t+1} &= (1-\delta_I) k_{It} + x_{It} \end{aligned}$$

where subscript T/I denotes tangible/intangible

- Technology 1 – producing goods and services

$$y_b = A^1 F(k_T^1, k_I, h^1)$$

- Technology 2 – producing intangible capital

$$x_I = A^2 G(k_T^2, k_I, h^2)$$

Total intangible stock used in two activities

Two Types of Intangible Investment

- Expensed: capital owners finance χ with reduced profits
- Sweat: worker owners finance $1-\chi$ with reduced wages

Choice of χ has tax implications

Hypothesis for the 1990s

- Technological change was nonneutral: $A_t^2/A_t^1 \uparrow$

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$\Rightarrow$ Measured productivity p_t^{NIPA} falls

$$p_t^{NIPA} \propto \frac{y_{bt}}{h_t^1 + h_t^2}$$

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$\Rightarrow$ Measured productivity p_t^{NIPA} falls

While true productivity p_t rises

$$p_t \propto \frac{y_{bt}}{h_t^1} = \frac{y_{bt} + q_t x_{It}}{h_t^1 + h_t^2}$$

- CKM's labor wedge, $1-\tau_{lt}$:

$$
\begin{aligned}
1-\tau_{lt} &= \psi \frac{1+\tau_{ct}}{1-\tau_{ht}} \cdot \frac{c_t}{y_{bt}} \cdot \frac{h_t}{1-h_t} \\
&= \psi \frac{1+\tau_{ct}}{1-\tau_{ht}} \cdot \frac{c_t}{y_{bt}} \cdot \frac{h_t^1}{1-h_t} \cdot \frac{h_t}{h_t^1} \\
&= 1+\frac{h_t^2}{h_t^1} \\
&= 1+\frac{q_t x_{It}}{y_{bt}}
\end{aligned}
$$

which is rising over the 1990s

Quantitative Predictions

Identifying TFPs

- Need inputs and outputs of production
  - Split of hours and tangible capital in 2 activities
  - Magnitude of intangible investment and capital

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  - Split of hours and tangible capital in 2 activities
  - Magnitude of intangible investment and capital

  $\Leftarrow$ Determined by factor price equalization

- Only requires observations on NIPA products and CPS hours

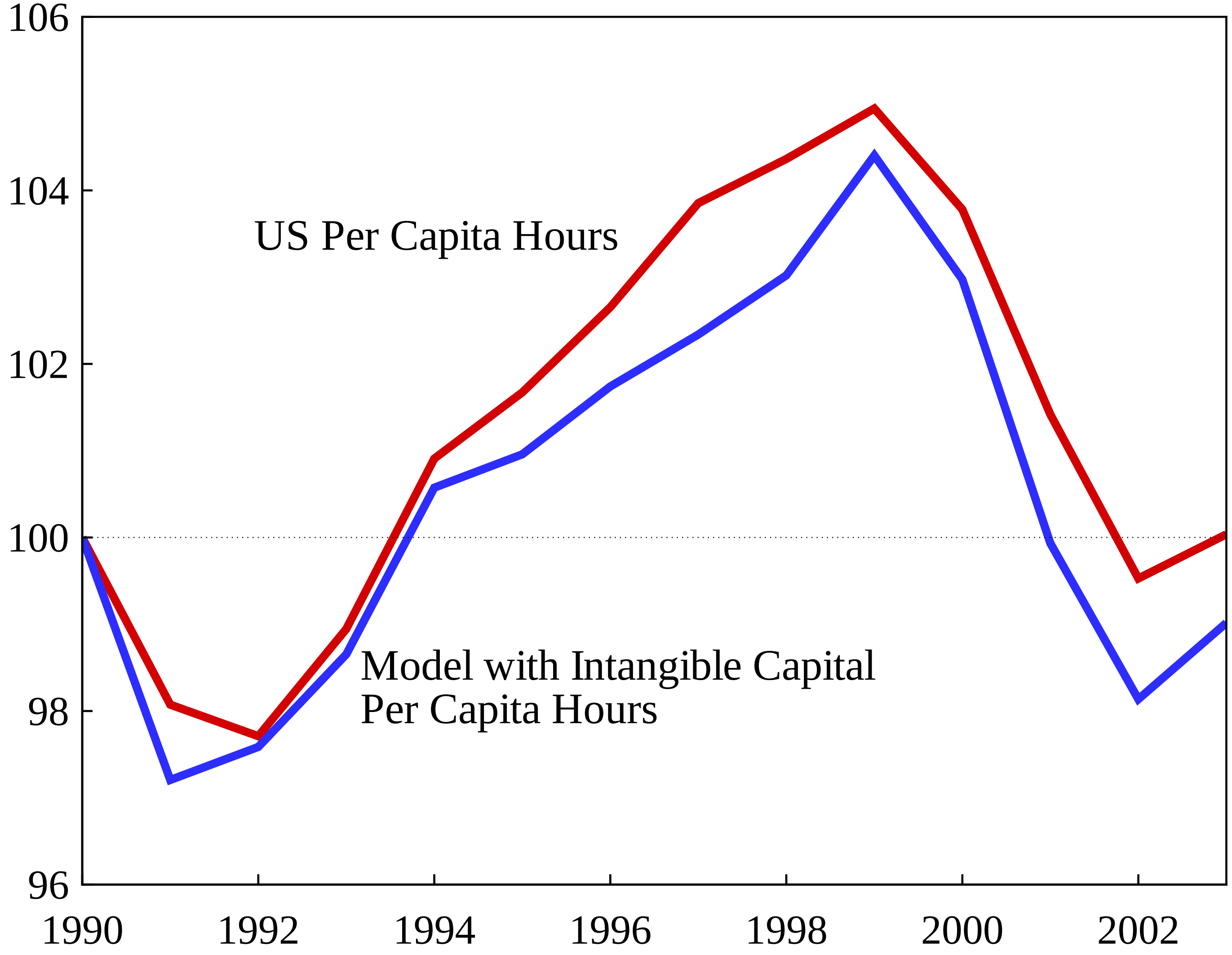
Compute Equilibrium Paths

- Computed both
  - Perfect foresight paths
  - Stochastic simulations

Results were insensitive to choice

Next, reconsider the prediction of per capita hours

Equilibrium Per Capita Hours



Downturn of 2008–2009

Downturn of 2008–2009

- Many who observed:
  - GDP and hours fall significantly
  - Labor productivity rise
- Concluded that this time is different

Downturn of 2008–2009

- Many who observed:
  - Rising credit spreads
  - Plummeting asset values
- Concluded financial market disruptions responsible

But, is this time different?

- 2008–2009 is “flip side” of 1990s:
  - GDP and hours depressed, but booming in ’90s
  - Labor productivity high, but low in ’90s
- In earlier work, found puzzling if abstract from
  - Intangible investment that is expensed
  - Nonneutral technology change w.r.t. its production

Application of Theory to 2000s

- Apply "off-the-shelf" model from 1990s study
  - Feed in paths for TFPs and tax rates
  - Abstract from financial and labor market disruptions

- Main findings:
  - Productivity growth slow-down big part of story
  - Aggregate observations in conformity with theory

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Is there any empirical evidence?

- *Intellectual property products* investment included:
  - R&D
  - Artistic originals
  - Software (first introduced in 1999)
- While much investment still missing, category is large...

BEA Comprehensive Revision 2013

- Private fixed nonresidential investment, 2012

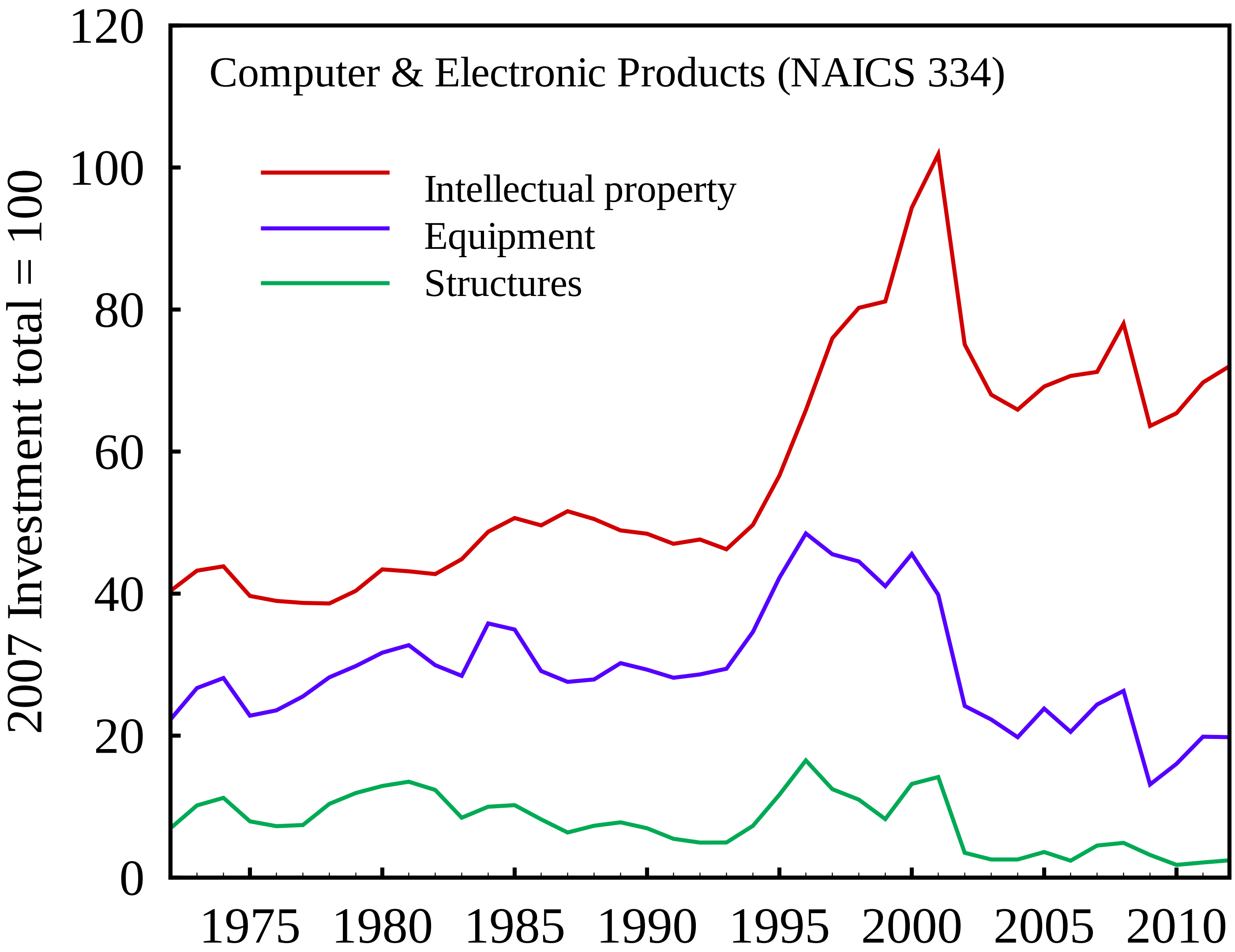
  22% Structures

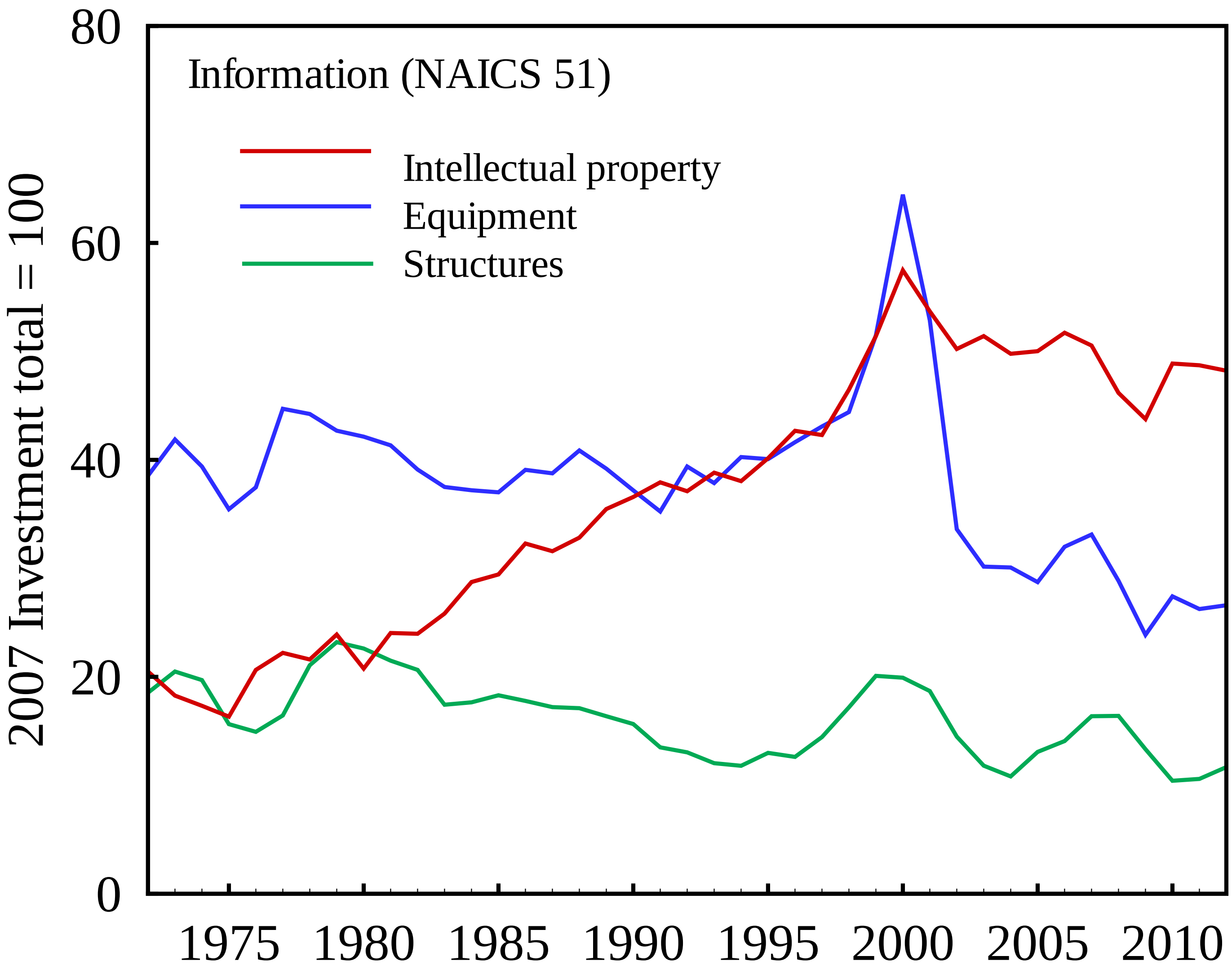
  45% Equipment

  33% Intellectual property

- Also have data for detailed industrial sectors

Computer & Electronic Products




Information (NAICS 51)
Intellectual property
Equipment
Structures
2007 Investment total = 100
80
60
40
20
0
1975
1980
1985
1990
1995
2000
2005
2010

Other Microevidence

- SEC requires 10-K reports from public companies
- Have company info on
  - R&D expenses
  - Advertising expenses
- Data show simultaneous large declines in 2008–2009

Top 500 Advertisers (COMPUSTAT)

Statistic	% of Domestic company total	% Decline in 2008–2009
Ad expenses	96.5	-10.8
R&D expenses	46.6	-16.2
PP&E expenses	27.5	-18.2
Employees	50.2	-2.2
Sales	38.6	-3.5

Top 500 R&D Spenders (COMPUSTAT)

Statistic	% of Domestic company total	% Decline in 2008–2009
Ad expenses	44.7	-19.6
R&D expenses	92.3	-11.9
PP&E expenses	25.9	-21.7
Employees	24.4	-4.4
Sales	34.2	-15.3

Strong I-O Linkages

- Use BEA's 2007 input-output benchmark
- Find 66% of output has intermediate uses from
  - Manufacturing (NAICS 31-33)
  - Information (NAICS 51)
  - Professional and business services (NAICS 54-56)
- And to sectors that do much less intangible investment

Recap

- Intangible investments are:
  - Expensed for tax purposes
  - Only partly measured in GDP
  - Estimated to be as large as tangibles
  - Correlated with tangibles
  - Picked up in typical productivity measures
- And, in our view, worthy of further investigation

Future research needed

- Need full exploration of microevidence for 2008-2009
- Main challenge is using theory to measure the unmeasured

Not everything that counts can be counted, and

not everything that can be counted counts.

— Albert Einstein

III. Back to methods: Nonlinearities and large state spaces

- Marimon, R. and A. Scott

  Computational Methods for the Study of Dynamic Economies

  Oxford University Press, 1999

  "Application of weighted residual methods to dynamic economic models"

- Finite element method has proven useful for:

  - Problems with nonlinearities (kinks, discontinuities)

  - Problems with large state spaces (exploits sparseness)

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Today, will describe method in context of Aiyagari & McGrattan

Aiyagari-McGrattan (1998)

- Study economies with
  - Large number of infinitely-lived households
  - Borrowing constraints
  - Precautionary savings motives

$\Rightarrow$ Savings decision functions have kinks

Distribution of asset holdings have discontinuities

Aiyagari-McGrattan (1998)

- Consumer problem:

$$\max_{\{\tilde{c}_t,\tilde{a}_{t+1},\ell_t\}} E\Big[\sum_{t=0}^{\infty}(\beta(1+g)^{\eta(1-\mu)})^t(\tilde{c}_t^{\eta}\ell_t^{1-\eta})^{1-\mu}/(1-\mu)|\tilde{a}_0,e_0\Big]$$

$$\text{s.t. } \tilde{c}_t+(1+g)\tilde{a}_{t+1}\leq(1+\bar{r})\tilde{a}_t+\bar{w}e_t(1-\ell_t)+\chi$$

$$\tilde{a}_t\geq 0$$

$$\ell_t\leq 1$$

$$e_t : \text{Markov chain}$$

where after-tax rates $\bar{w}$, $\bar{r}$ and transfers χ given

- Consumer problem:

$$\max_{\{\tilde{c}_t,\tilde{a}_{t+1},\ell_t\}} E\Big[\sum_{t=0}^{\infty}(\beta(1+g)^{\eta(1-\mu)})^t(\tilde{c}_t^{\eta}\ell_t^{1-\eta})^{1-\mu}/(1-\mu)|\tilde{a}_0,e_0\Big]$$

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$$\tilde{a}_t\geq 0$$

$$\ell_t\leq 1$$

$$e_t : \text{Markov chain}$$

where after-tax rates $\bar{w}$, $\bar{r}$ and transfers χ given

Restrictions on $a_t, e_t \Rightarrow$ kinks, discontinuities

Let's get rid of constraints

- Modified objective ($\tilde{\beta} = \beta(1+g)^{\eta(1-\mu)}$):

$$E\left[\sum_{t=0}^{\infty} \tilde{\beta}^t \left\{ \frac{(\tilde{c}_t^{\eta} \ell_t^{1-\eta})^{1-\mu}}{1-\mu} + \frac{\zeta}{3}(\min(\tilde{a}_t, 0)^3 + \min(1-\ell_t, 0)^3) \right\} | \tilde{a}_0, e_0 \right]$$

- Solve a sequence of problems with $\zeta = 1, 10, 100$, etc.

- Want: functions $c(x,i)$, $\ell(x,i)$, $\alpha(x,i) = \tilde{a}'$ that solve FOCs

Really only need to find $\alpha(x, i)$

- $c(x, i)$ from budget constraint given $\alpha(x, i)$

- $\ell(x, i)$ from intratemporal condition given $\alpha(x, i)$

Note: in case of ℓ need a robust Newton routine

Boils down to...

- Find $\alpha(x,i)$ to set $R(x,i;\alpha)=0$:

$$R(x,i;\alpha) = \eta(1+g)c(\ell^*(x,i;\alpha))^{\eta(1-\mu)-1}\ell^*(x,i;\alpha)^{(1-\eta)(1-\mu)} \\ - \beta(1+g)^{\eta(1-\mu)}\{\sum_j \pi_{i,j}\,\eta(1+\bar{r})c(\ell^*(\alpha(x,i),j;\alpha))^{\eta(1-\mu)-1} \\ \cdot\,\ell^*(\alpha(x,i),j;\alpha)^{(1-\eta)(1-\mu)} + \zeta\min(\alpha(x,i),0)^2\},$$

where $c^*(x,i;\alpha)$, $\ell^*(x,i;\alpha)$ from static FOCs

Applying the Finite Element Method

- Find $\alpha^h(x,i)$ to set $R(x,i;\alpha^h) \approx 0$
- Steps (for Galerkin variant with linear bases):
  1. Partition $[0, x_{max}]$, with subintervals called elements
  2. Define α^h on $[x_e, x_{e+1}]$:
  $$\alpha^h(x,i) = \psi_e^i N_e(x) + \psi_{e+1}^i N_{e+1}(x)$$
  $$N_e(x) = \frac{x_{e+1} - x}{x_{e+1} - x_e}, \; N_{e+1}(x) = \frac{x - x_e}{x_{e+1} - x_e}$$
  3. Find ψ_e^i's to satisfy
  $$F(\vec{\psi}) = \int R(x,i;\alpha^h) N_e(x) dx = 0, \; i = 1, \ldots, m, \; e = 1, \ldots n.$$

$\Rightarrow$ Solve mn nonlinear equations in mn unknowns

Practicalities

- It helps to...
  - Adapt the grid to optimally partition the grid
  - Compute analytical derivatives $dF/d\psi_e^i$ to get speed
  - Exploit sparseness of jacobian matrix

Computing the Invariant Distribution

- Want equilibrium prices r, w

- Need $H(x,i) = Pr(x_t < x \,|\, e_t = e(i))$ which solves:

$$H(x,i) = \sum_{j=1}^{m} \pi_{j,i} H(\alpha^{-1}(x,j),j) I(x \geq \alpha(0,j)), I(x > y) = 1 \text{ if } x > y$$

- Can again apply FEM to this

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- Can again apply FEM to this
- How well does it work given the kinks and discontinuities?

Test Cases with Known Solutions

- For test of $\alpha(x, i)$ computation, assume labor inelastic and $e_t = 1$
- For test of $H(x, i)$ computation,
  - Make up a tractible $\alpha(x, i)$ that
  - Generates known invariant distribution

Testing $\alpha(x, i)$

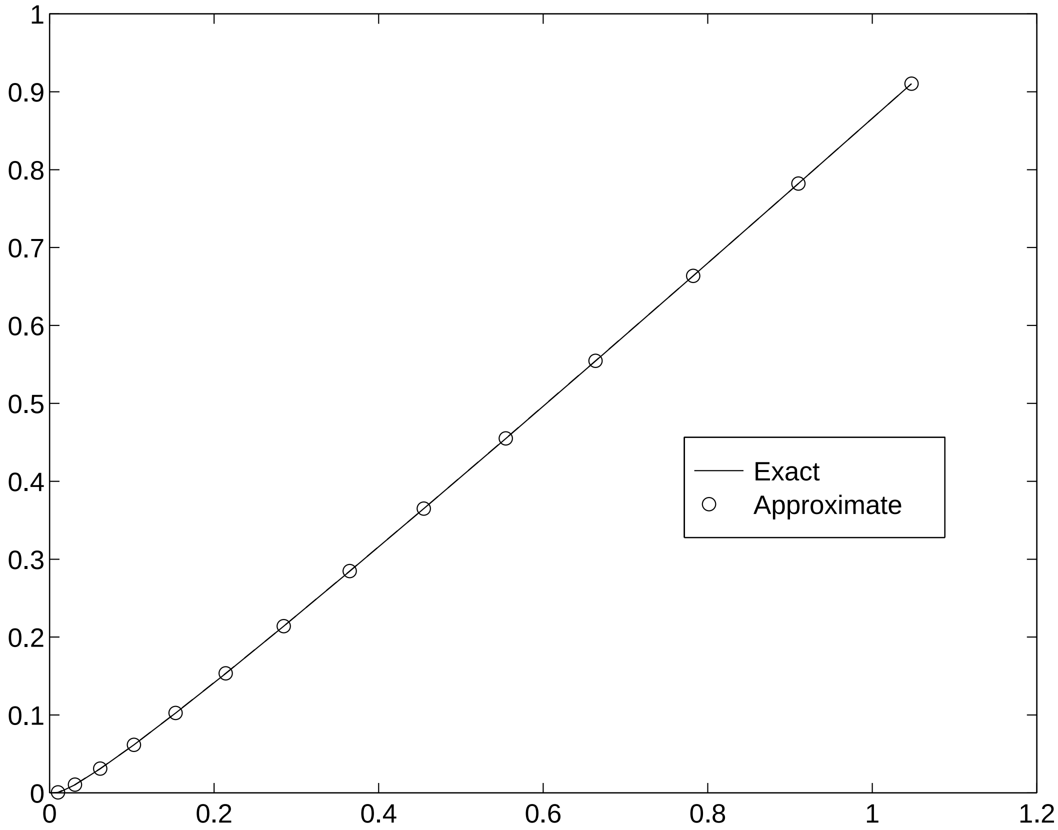
- Consumer problem:

$$\max_{\{c_t, a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

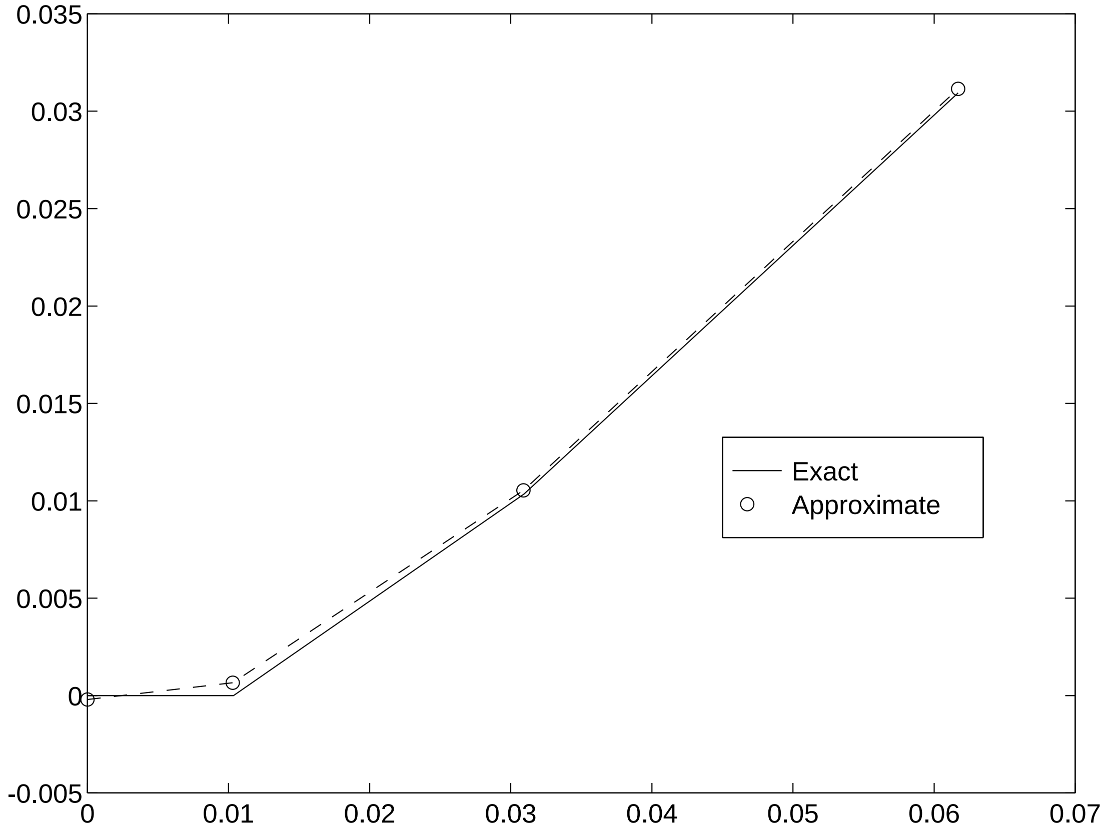
$$\text{subject to} \quad c_t + a_{t+1} = (1+r)a_t + w$$

- Solution is piecwise linear and analytically computed

Testing $\alpha(x, i)$–Grid known

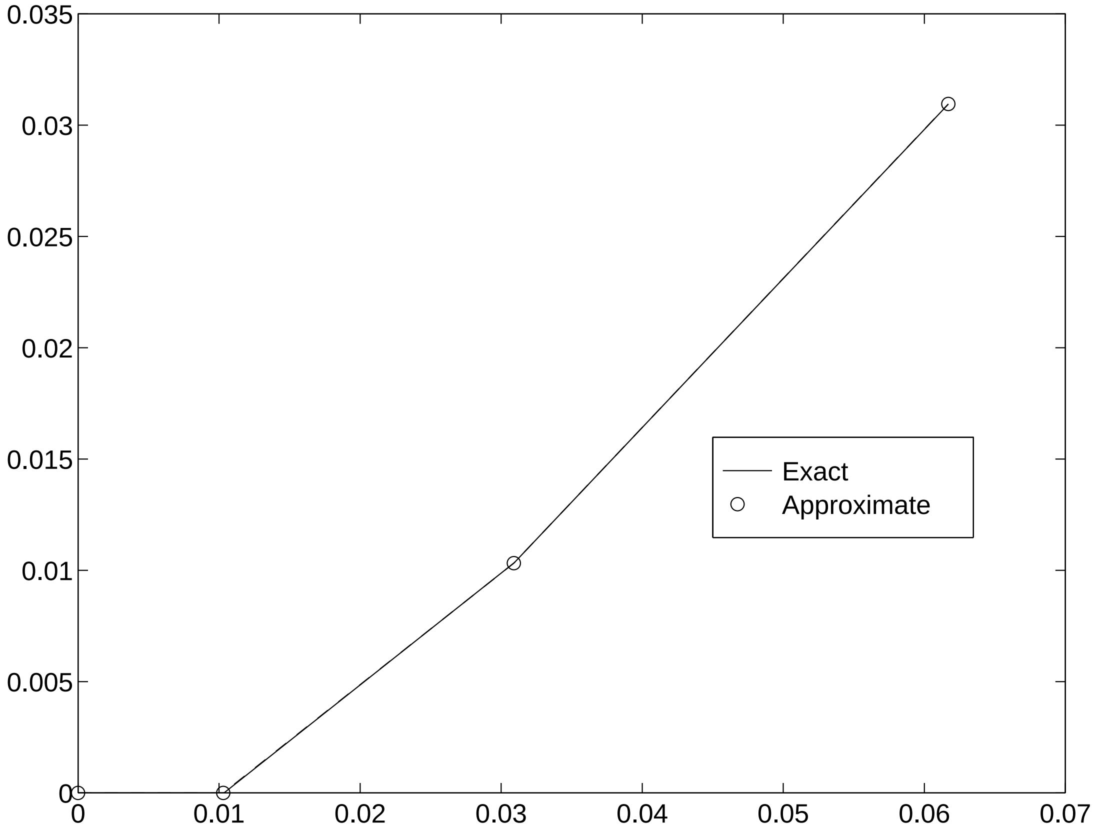


Testing $\alpha(x,i)$–Grid known



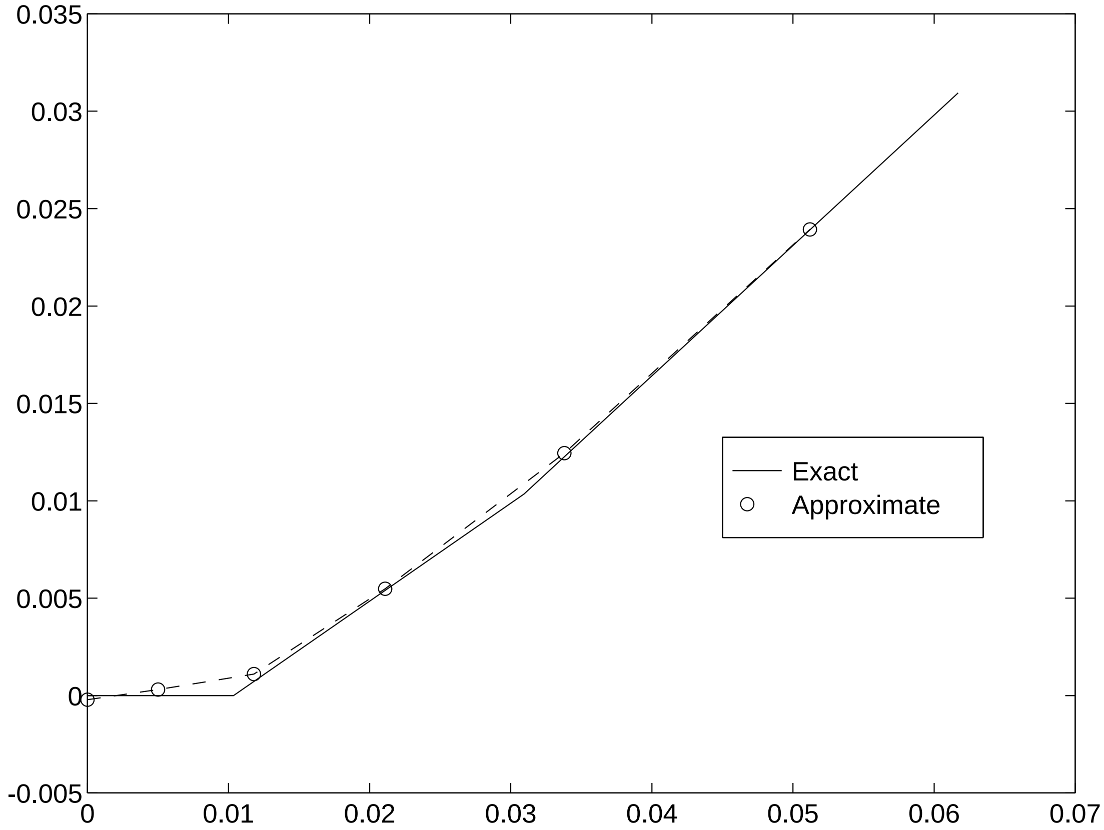
Zoomed in, Boundary conditions not imposed

Testing $\alpha(x,i)$–Grid known



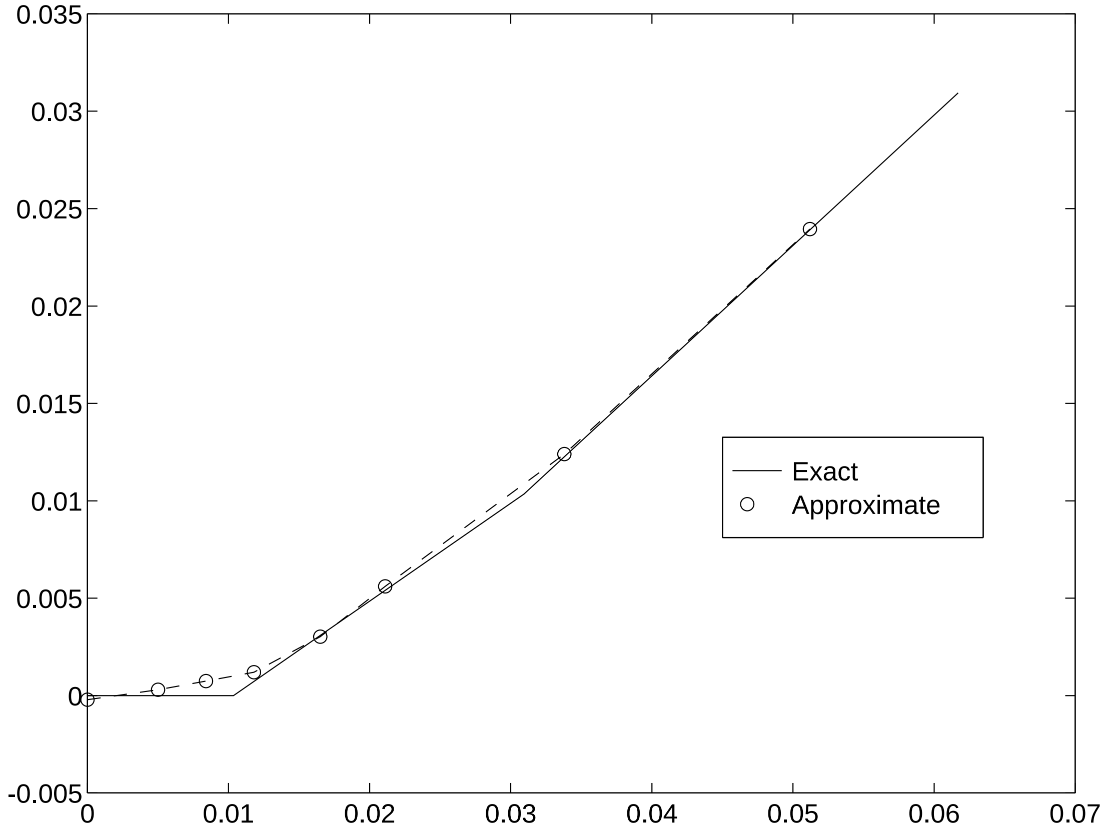
Zoomed in, Boundary conditions imposed

Testing $\alpha(x, i)$–Grid not known



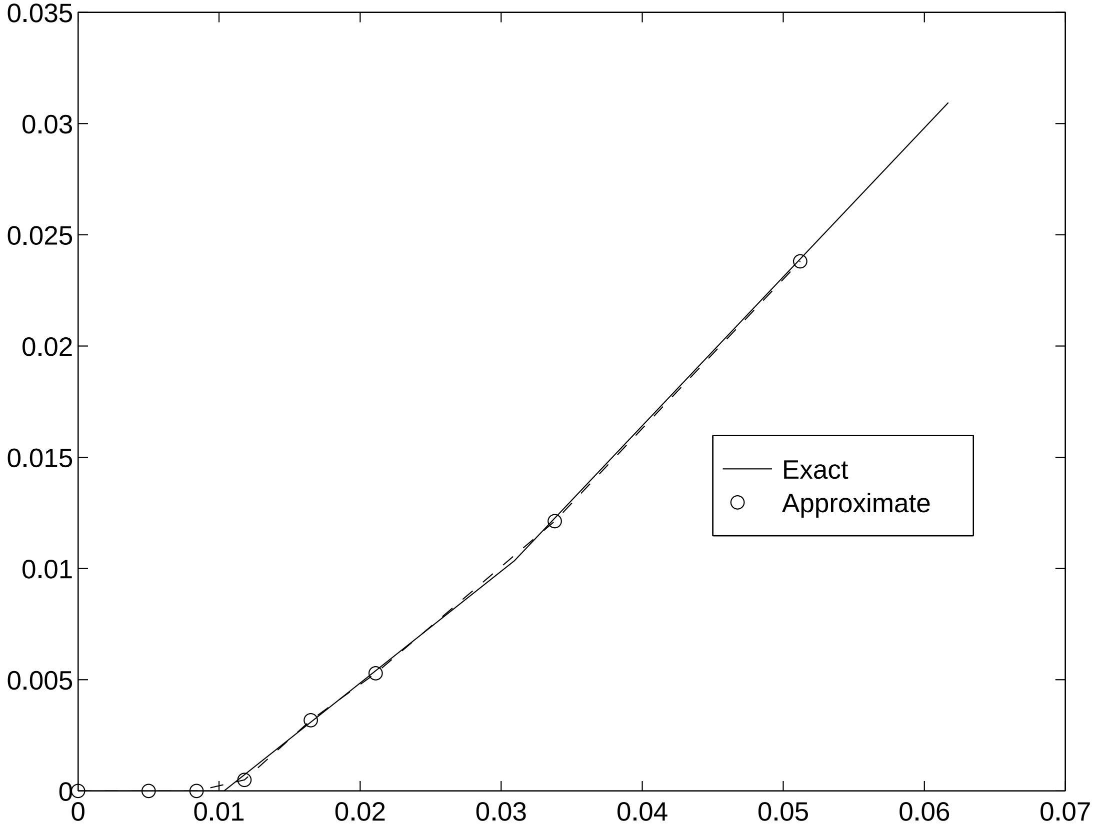
Zoomed in, Boundary conditions not imposed

Testing $\alpha(x,i)$–Grid not known



Zoomed in, Boundary conditions not imposed, Adapt grid

Testing $\alpha(x, i)$–Grid not known



Zoomed in, Boundary conditions imposed, Adapt grid

Testing $H(x, i)$

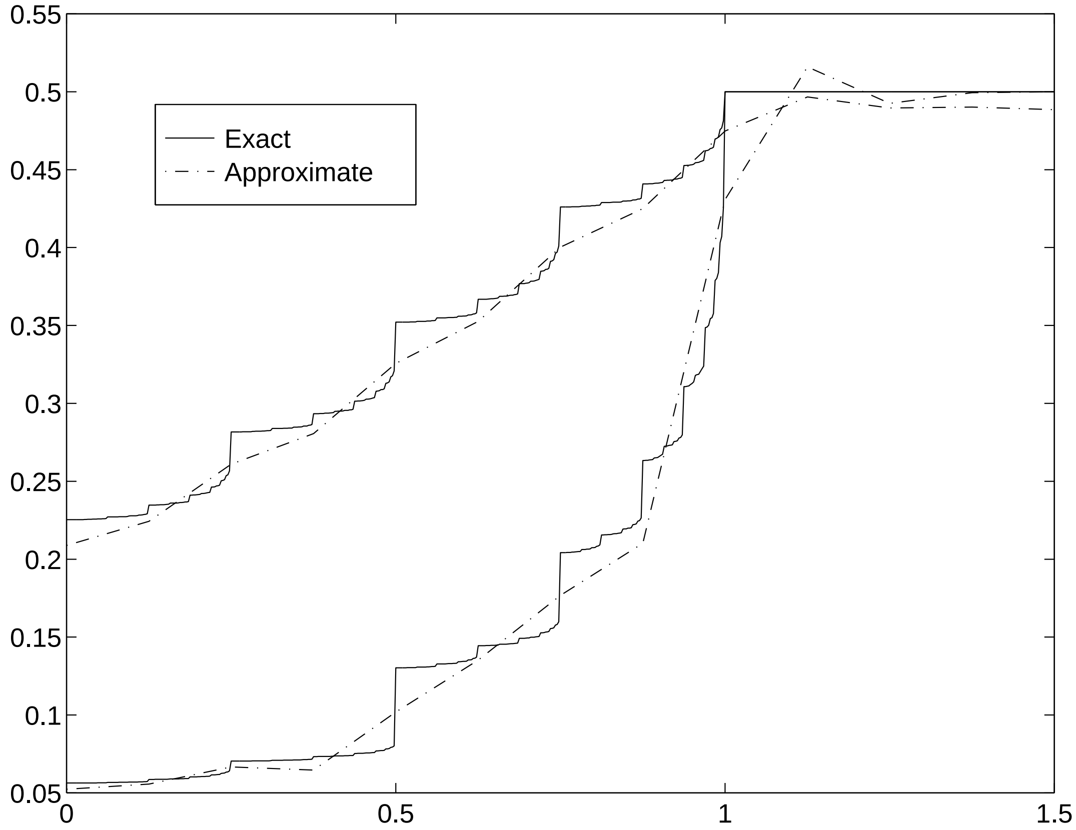
- Suppose $\alpha(x, i)$ is:

$$\alpha(x,i) = \begin{cases} \max(0, -0.25 + x), & \text{if } i = 1 \\ 0.5 + 0.5x, & \text{if } i = 2, \end{cases}$$

  with $\pi_{1,1} = \pi_{2,2} = 0.8$.

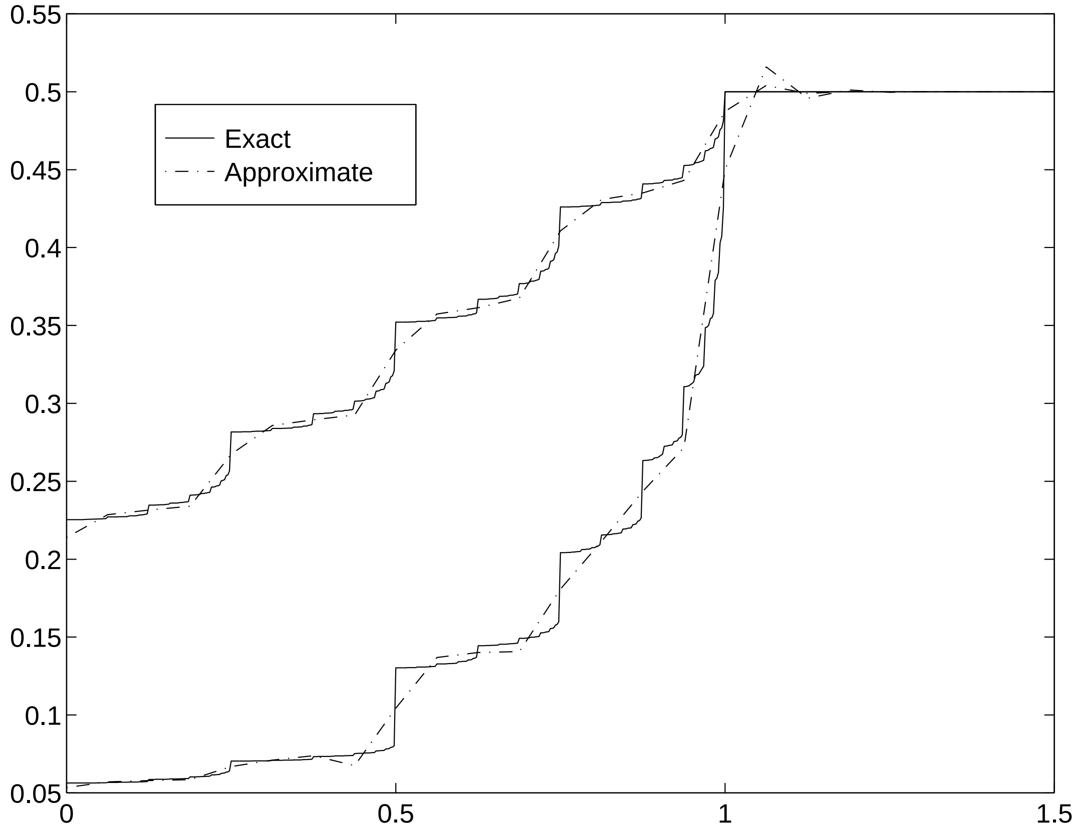
- Then, it is easy to analytically derive H

Testing $H(x,i)$—Evenly spaced grid



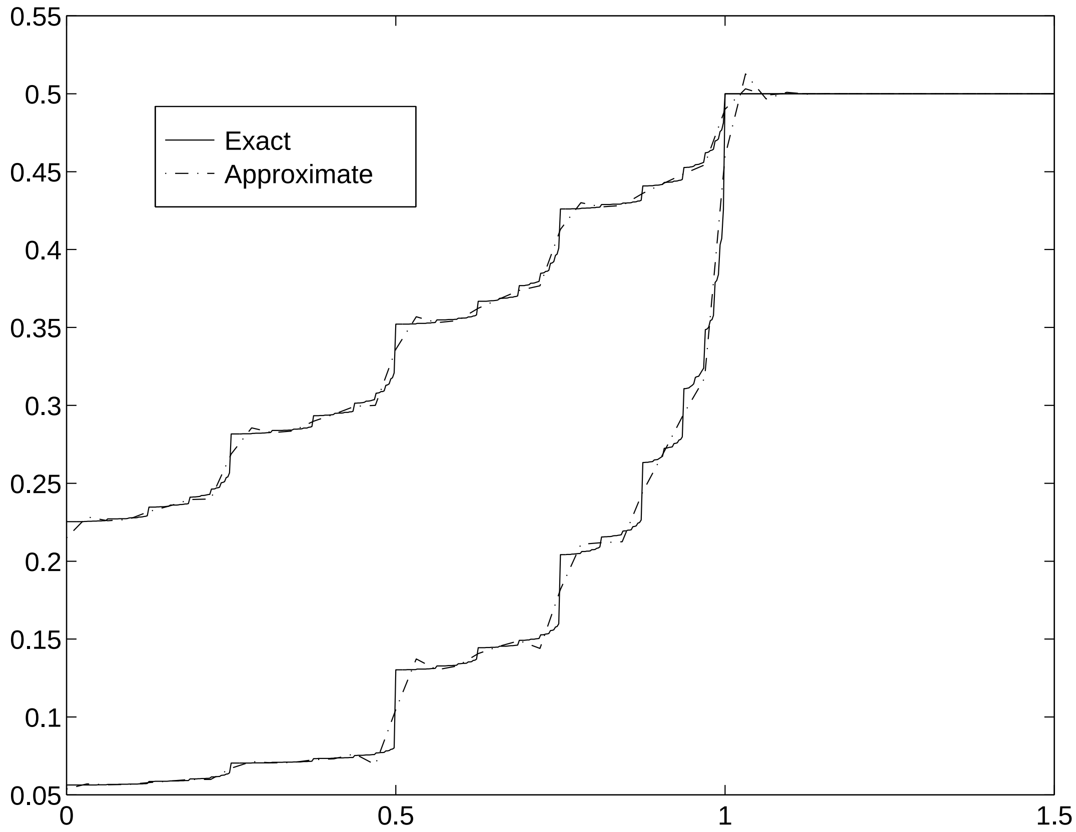
Have $n = 13$ elements

Testing $H(x,i)$—Evenly spaced grid



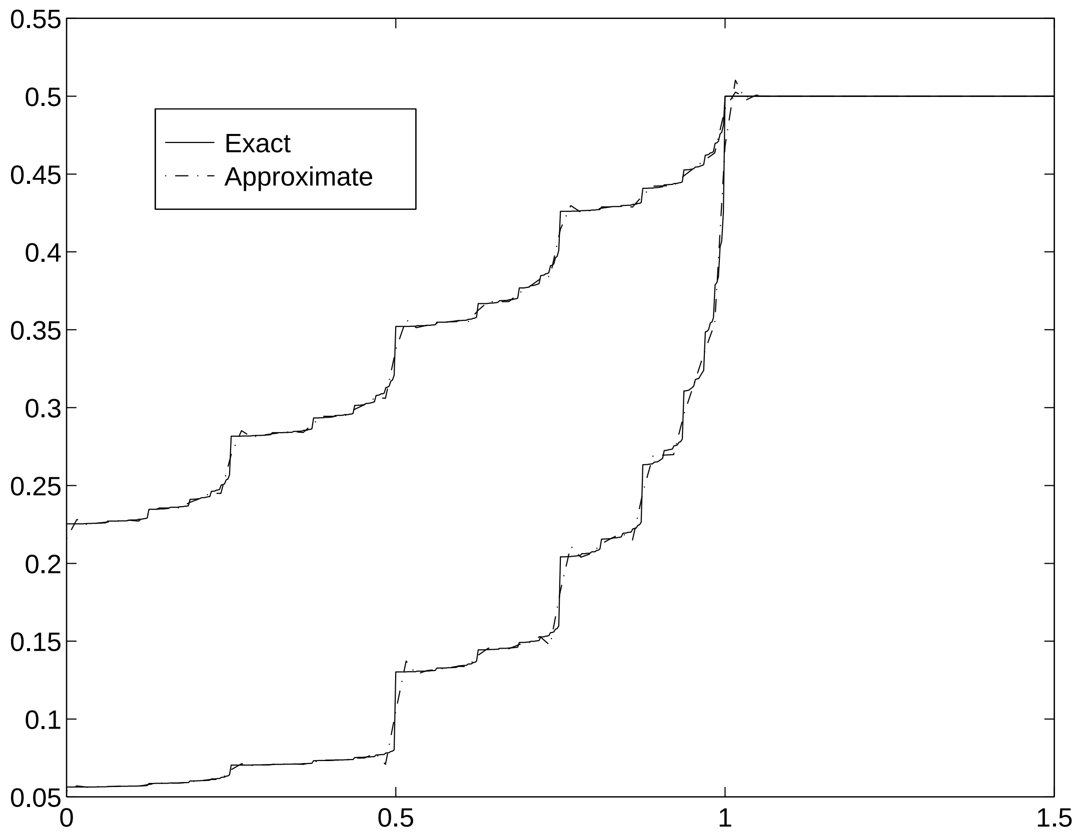
Add more elements ($n = 25$)

Testing $H(x,i)$—Evenly spaced grid



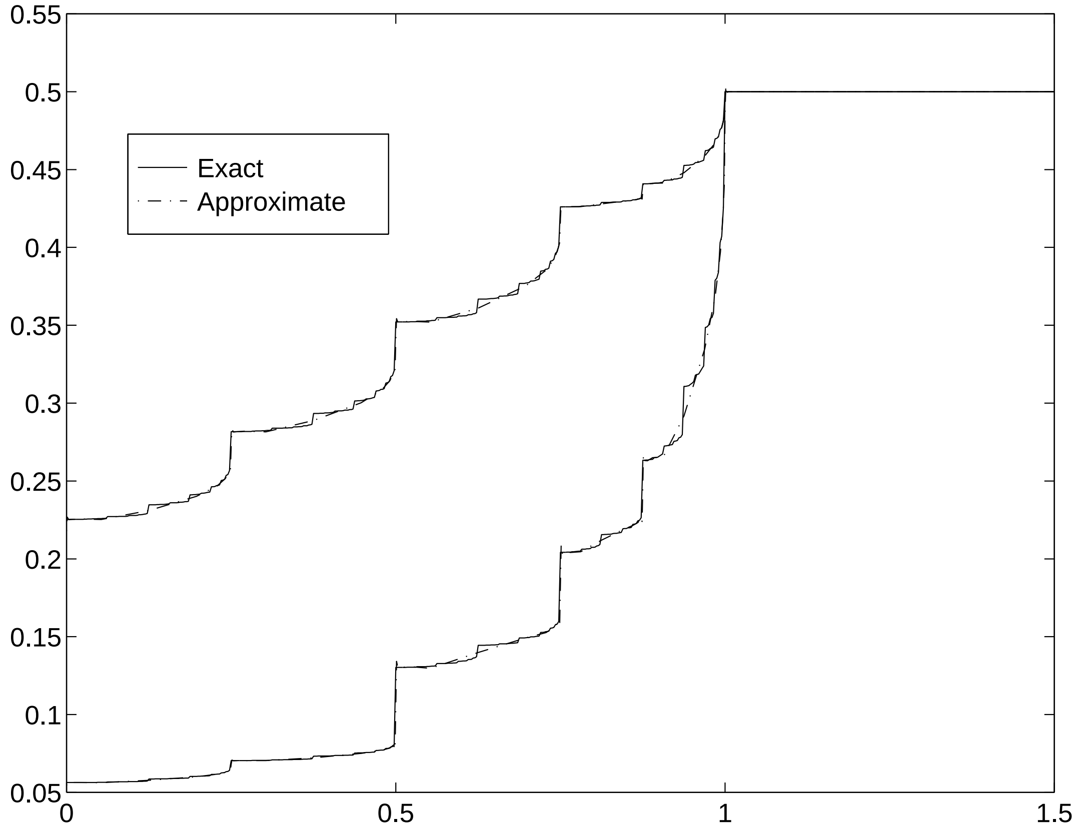
Add more elements ($n = 49$)

Testing $H(x, i)$—Evenly spaced grid



Add more elements ($n = 97$)

Testing $H(x, i)$—Not evenly spaced grid



Adapted grid ($n = 73$)

Future work needed

- Want to solve problems with time-varying distributions H_t

Parallel processing

- Big change since EUI Lectures in 1996: parallel processing
    - Most problems can be parallelized
    - OpenMPI simple to use with few changes to existing codes

Recap of Lecture III

- Since 1996, I have
  - Applied FEM to many interesting problems
  - Learned to parallelize most of my codes
- But, there is still much to learn!