Notes on Dixit-Stiglitz Size Distribution Model
Econ 8601

1. Model

Consider the following partial equilibrium model of an industry. The final good in the industry is a composite of differentiated products.

Let x be a type of good. Let $q(\cdot)$ map from $R^+ \rightarrow R^+$ be a quantity of differentiated good x. Then the amount of the final good is

$$Q = F(q) = \left(\int_0^\infty w(x) q(x)^{\frac{1}{\mu}} dx \right)^\mu$$

where $w(x) > 0$, $w' \leq 0$. The elasticity of substitution is

$$\sigma = \frac{\mu}{\mu - 1}.$$

Assume that $\mu > 1$ (which is equivalent to $\sigma > 1$). (Without this assumption demand for differentiated products is inelastic which causes problems with analyzing a profit maximizing monopolist).

Demand for the final good is given by,

$$D(P) = Q^d.$$

Assume this is strictly decreasing in the composite price P.

Opening a plant to produce a particular differentiated product x costs ϕ dollars. There is a constant marginal cost of $c > 0$ dollars.

There are three stages in the model.

(1) Potential entrants simultaneously decide whether or not to enter and pick a product x to produce and pay the fixed cost $\phi > 0$. Let $E(x) \in \{0, 1, 2, ...\}$ be the integer number of firms that enter to produce product x.

(2) Each entrant i producing product x simultaneously choose prices $p_i(x)$.

(3) Consumers observe prices of the differentiated inputs and make consumption decisions. Let $q_i(x)$ be the demand of entrant i.

2. Equilibrium

We restrict attention to subgame perfect equilibria. The following conditions must be satisfied:

(1) Entry decisions are profit maximizing. Any firms that enter make nonnegative profit. There is no incentive for additional firm entry. In particular, if an additional firm enters, its profit in the ensuing subgame beginning at stage 2 must be no greater than the fixed cost ϕ of entry.

(2) Firms set prices to maximize profits, taking as given the prices set by the other firms and the demand behavior of consumers.

(3) For each unit of the composite consumed, consumers assemble the unit in the cost minimizing way. The composite price P is the cost of this expenditure minimizing bundle. The total demand for the composite equals $Q = D(P)$.

It is immediate that if $E(x) = 2$ for some x, the equilibrium price for the two firms in stage 2 must be marginal cost, $p_1(x) = p_2(x) = c$, so both firms lost the fixed cost ϕ. Thus $E(x) = 2$ cannot happen in equilibrium. Analogously, we can rule out $E(x) \geq 3$. Thus we restrict attention to $E(x) \in \{0, 1\}$.

3. Characterization of Equilibrium

3.1 The Consumer Problem

Consider the consumer expenditure minimization problem for constructing one unit of composite. Let P be the cost of the expenditure minimizing bundle,

$$\begin{aligned} P &= \min_{\overline{q}} \int_0^\infty p(x)\overline{q}(x)dx \\ \text{such that} &: \left(\int_0^\infty w(x)\overline{q}(x)^{\frac{1}{\mu}} dx \right)^\mu = 1 \end{aligned}$$

Consider two particular products x_0 and x_1. The MRS is

$$\begin{aligned} \frac{\frac{\partial Q}{\partial q_0}}{\frac{\partial Q}{\partial q_1}} &= \frac{\mu \left(\int_0^\infty w(x)\overline{q}(x)^{\frac{1}{\mu}} dx \right)^{\mu-1} w_0 \frac{1}{\mu} \overline{q}_0^{\frac{1}{\mu}-1}}{\mu \left(\int_0^\infty w(x)\overline{q}(x)^{\frac{1}{\mu}} dx \right)^{\mu-1} w_1 \frac{1}{\mu} \overline{q}_1^{\frac{1}{\mu}-1}} \\ &= \frac{w_0 \overline{q}_0^{\frac{1-\mu}{\mu}}}{w_1 \overline{q}_1^{\frac{1-\mu}{\mu}}} \\ &= \frac{w_0 \overline{q}_0^{-\frac{1}{\sigma}}}{w_1 \overline{q}_1^{-\frac{1}{\sigma}}} \end{aligned}$$

Expenditure minimization implies

$$\begin{aligned} \frac{w_0 \overline{q}_0^{-\frac{1}{\sigma}}}{w_1 \overline{q}_1^{-\frac{1}{\sigma}}} &= \frac{p_0}{p_1} \\ \frac{\overline{q}_0}{\overline{q}_1} &= \left(\frac{w_1}{w_0} \frac{p_0}{p_1} \right)^{-\sigma} \end{aligned}$$

Or

$$\overline{q}_1 = \overline{q}_0 \left(\frac{w_0}{w_1} \frac{p_1}{p_0} \right)^{-\sigma}.$$

Substitute this into the constraint,

$$
\begin{aligned}
1 &= \\
&= \left(\int_0^\infty w(x)\overline{q}(x)^{\frac{1}{\mu}}dx\right)^\mu \\
&= \left(\int_0^\infty w(x)\left(\overline{q}_0\left(\frac{w(0)}{w(x)}\frac{p(x)}{p(0)}\right)^{-\sigma}\right)^{\frac{1}{\mu}}dx\right)^\mu \\
&= \overline{q}_0\left(\int_0^\infty w(x)\left(\left(\frac{w(0)}{w(x)}\frac{p(x)}{p(0)}\right)^{-\sigma}\right)^{\frac{1}{\mu}}dx\right)^\mu
\end{aligned}
$$

$$
\begin{aligned}
\overline{q}_0 &= \left(\int_0^\infty w(x)\left(\left(\frac{w(0)}{w(x)}\frac{p(x)}{p(0)}\right)^{-\sigma}\right)^{\frac{1}{\mu}}dx\right)^{-\mu} \\
&= \left(\frac{w(0)}{p(0)}\right)^\sigma\left(\int_0^\infty w(x)^\sigma p(x)^{1-\sigma}dx\right)^{-\frac{\sigma}{\sigma-1}}
\end{aligned}
$$

The minimum cost is

$$
\begin{aligned}
P = &\int_0^\infty p(x)\overline{q}(x)dx \\
= &\int_0^\infty p(x)\overline{q}_0\left(\frac{w(0)}{w(x)}\frac{p(x)}{p(0)}\right)^{-\sigma}dx \\
= &\ \overline{q}_0\left(\frac{w(0)}{p(0)}\right)^{-\sigma}\int_0^\infty p(x)^{1-\sigma}w(x)^\sigma dx \\
= &\left(\int_0^\infty w(x)^{1-\frac{\sigma}{\mu}}p(x)^{-\frac{\sigma}{\mu}}dx\right)^{-\mu}\int_0^\infty p(x)^{1-\sigma}w(x)^\sigma dx \\
= &\left(\int_0^\infty w(x)^\sigma p(x)^{1-\sigma}dx\right)^{-\frac{\sigma}{\sigma-1}}\int_0^\infty p(x)^{1-\sigma}w(x)^\sigma dx \\
= &\left(\int_0^\infty w(x)^\sigma p(x)^{1-\sigma}dx\right)^{-\frac{1}{\sigma-1}}
\end{aligned}
$$

So we can write $\overline{q}_0$ as

$$
\overline{q}_0 = \left(\frac{w(0)}{p(0)}\right)^\sigma P^\sigma
$$

$$\overline{q}_1 = \overline{q}_0 \left(\frac{w_0}{w_1} \frac{p_1}{p_0} \right)^{-\sigma} \tag{1}$$

We can now derive the demand function faced by an individual firm.

$$\begin{aligned} q_1 &= Q\overline{q}_1 \\ &= D(P)\overline{q}_0 \left(\frac{w_0}{w_1} \frac{p_1}{p_0} \right)^{-\sigma} \\ &= D(P)\overline{q}_0 \left(\frac{w_0}{w_1 p_0} \right)^{-\sigma} p_1^{-\sigma} \\ &= \kappa_1 p_1^{-\sigma} \end{aligned}$$

Note from the firm's perspective κ_1 does not depend upon its choice of p_1, since the firm has measure zero, p_1 does not affect $\overline{q}_0$ or P.

3.2 The Profit-Maximizing Price

Consider the price-setting problem of the firm producing a particular good x_1. As argued above, we know that the firm has a monopoly on the good. The problem of the firm is

$$\begin{aligned} & \max_p q_1(p_1) (p_1 - c) - \phi \\ = \; & \max_p \kappa_1 p_1^{-\sigma} (p_1 - c) - \phi \end{aligned}$$

The FONC is

$$-\sigma \kappa_1 p_1^{-\sigma-1} (p_1 - c) + \kappa_1 p_1^{-\sigma} = 0,$$

or

$$
\begin{aligned}
\sigma p_1^{-1}(p_1 - c) &= 1 \\
\frac{p_1 - c}{p_1} &= \frac{1}{\sigma} \\
p_1 &= \mu c
\end{aligned}
$$

Note that all firms set the same price.

3.3 An Observation about Entry

It is immediate that since $w(x)$ is nonincreasing, firm profit must be nonincreasing as a function of x. Therefore, we can restrict attention to entry that takes the form of a cutoff rule n such that $E(x) = 1$ for $x \leq n$ and $E(x) = 0$ for $x > n$.

3.4 The Composite Price

Take as given that the measure of entry is n. Since each firm sets the same price $p(x) = \mu c$. The composite price can be calculated.

$$
\begin{aligned}
P &= \left(\int_0^n w(x)^\sigma (\mu c)^{1-\sigma}\, dx \right)^{-\frac{1}{\sigma-1}} \\
&= \left(\int_0^n w(x)^\sigma dx \right)^{-\frac{1}{\sigma-1}} \mu c
\end{aligned}
$$

We can also solve for the amount of good 0 used to construct the unit composite

$$
\begin{aligned}
\overline{q}_0 &= \left(\frac{w(0)}{\mu c} \right)^\sigma \left(\int_0^n w(x)^\sigma (\mu c)^{1-\sigma}\, dx \right)^{-\frac{\sigma}{\sigma-1}} \\
&= w(0)^\sigma \left(\int_0^n w(x)^\sigma dx \right)^{-\frac{\sigma}{\sigma-1}}
\end{aligned}
$$

From (1) we can calculate the amount of any good x used to construct the unit composite,

$$\begin{aligned} \overline{q}(x) &= \overline{q}_0 \left(\frac{w(0)}{w(x)} \right)^{-\sigma} && (2) \\ &= w(x)^{\sigma} \left(\int_0^n w(x)^{\sigma} dx \right)^{-\frac{\sigma}{\sigma-1}} && (3) \end{aligned}$$

Consider the special case where $w(x) = 1$. Here the composite price reduces to

$$P = n^{-\frac{1}{\sigma-1}} \mu c,$$

and the output in the unit composite is

$$\overline{q} = n^{-\frac{\sigma}{\sigma-1}}$$

3.5 Entry

The profit of firm x given total entry n is

$$\begin{aligned} \pi(x,n) &= (\mu - 1)\, cq(x) - \phi \\ &= (\mu - 1)\, cw(x)^{\sigma} \left(\int_0^n w(x)^{\sigma} dx \right)^{-\frac{\sigma}{\sigma-1}} D(P) - \phi \\ &= (\mu - 1)\, cw(x)^{\sigma} \left(\int_0^n w(x)^{\sigma} dx \right)^{-\frac{\sigma}{\sigma-1}} D\left(n^{-\frac{1}{\sigma-1}} \mu c \right) - \phi \end{aligned}$$

An equilibrium is where $\pi(n,n) = 0$.

Let $\tilde{\pi}(n) = \pi(n,n)$. Make the following assumption:

(Assumption 1). The parameters (i.e. σ, ϕ, c, and the $w(\cdot)$ and $D(\cdot)$ functions) are such that $\tilde{\pi}(n)$ is strictly decreasing in n. Furthermore,

To understand the contents of this assumption, consider the special case where $D(p) =$

$p^{-\eta}$ and $w(x) = 1$. Then

$$\begin{aligned}\tilde{\pi}(n) &= (\mu - 1)\, n^{-\frac{\sigma}{\sigma-1}} \left(n^{-\frac{1}{\sigma-1}}\right)^{-\eta} (\mu c)^{-\eta} - \phi \\ &= (\mu - 1)\, n^{\frac{\eta-\sigma}{\sigma-1}} (\mu c)^{-\eta} - \phi\end{aligned}$$

So here the assumption is equivalent so the assumption that $\eta < \sigma$. This is plausible, since the substitution possibilities are thought to be greater within differentiated inputs of the same commodity, then across different composite commodities.

Under Assumption 1, there is a unique equilibrium entry level $\hat{n}$ where

$$\tilde{\pi}(\hat{n}) = 0.$$

4. Discussion of the Size Distribution

In the equilibrium, the size of firm x relative to firm 0 is

$$\begin{aligned}q(x) &= q(0) \left(\frac{w(0)}{w(x)}\right)^{-\sigma} \\ & \quad q(0) w(0)^{-\sigma} w(x)^{\sigma}\end{aligned}$$

which decreases in x. Suppose we observe the size distribution in an industry given by $q(\cdot)$, ($q(x)$ is the size of firm x ranked by decreasing size). Then given a value for σ, we can backout what $w(x)$ is from the above.

Consider the effect of an increase in demand on the size distribution. Let λ parameterize demand and let $D(P, \lambda) = \lambda H(P)$, so λ is a proportional shift.

Case $w(x) = 1$.

Let q be firm size. We have at the equilibrium n,

$$\begin{aligned}(\mu - 1)\, cq - \phi &= 0 \\ q &= \frac{\phi}{(\mu - 1)\, c}\end{aligned}$$

So size is fixed. An increase in demand is met entirely by a change on the extensive margin (the addition of new firms.

Case $w(x)$ strictly decreasing.

Then

$$\tilde{\pi}(n) = (\mu - 1)\, cw(x)^{\sigma} \left(\int_0^n w(x)^{\sigma} dx \right)^{-\frac{\sigma}{\sigma - 1}} \lambda H(n^{-\frac{1}{\sigma-1}} \mu c) = \phi$$

Suppose we increase λ. Note the size of the marginal entrant remains fixed at $\hat{q}$ solving,

$$q = \frac{\phi}{(\mu - 1)\, c}$$

Firms already in the industry grow larger with λ, and new firms are added. The range of firm sizes in the industry increases (since the $x = 0$ firm grows larger).